MATH1510 Calculus for Engineers 2019-20

§ 1 Preliminaries

1.1 Notations

Set : collection of objects (elements)

$\subseteq$: subset

$\in$: belongs to

Example 1.1.1

$S = \{1, 2, 3\}$

That means S is a set containing 3 elements, namely 1, 2 and 3.

OR : $1, 2, 3 \in S$

If $T = \{1, 2, 3, 4\}$, then we say S is a subset of T, or $S \subseteq T$.

That means every element in S is also an element in T.

Notations often used in this course :

$\mathbb{Z}^+$: set of all positive integers

$\mathbb{Z}$: set of all integers

$\mathbb{R}$: set of all real numbers

ϕ : empty set, i.e. $\phi = \{\ \}$ Nothing

$[a,b]$: set of all real numbers x such that $a \leq x \leq b$

(a,b) : set of all real numbers x such that $a < x < b$

$[a,\infty)$: set of all real numbers x such that $a \leq x$

Example 1.1.2

Set of all positive even numbers

$= \{2, 4, 6, \ldots\}$

$= \{2m : m \in \mathbb{Z}^+\}$

i.e. this set consists of elements of the form $2m$ such that $m \in \mathbb{Z}^+$.

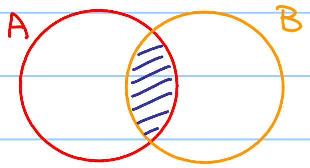
Exercise 1.1.1

Set of all positive odd numbers = ? (How to describe?)

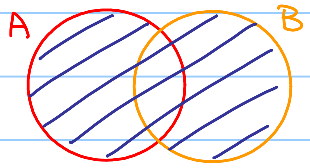
Answer : $\{2m-1 : m \in \mathbb{Z}^+\}$

Set Operations :

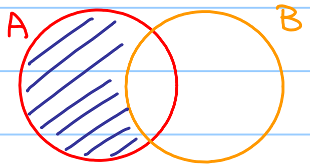
Let A, B be two sets.



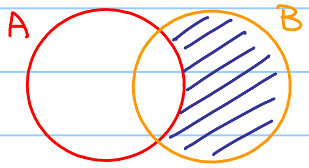
Intersection : $A \cap B$



Union : $A \cup B$



Relative complement of B in A : $A \setminus B$



Relative complement of A in B : $B \setminus A$

Example 1.1.3

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3\}$

- $A \cap B = \{2\}$ $\quad A \cap C = \phi$
- $A \cup B = \{1,2,3\}$
- $A \setminus B = \{1\}$ $\quad B \setminus A = \{3\}$

Example 1.1.4

$\mathbb{R} \setminus \{2\}$: set of all real numbers except 2

(Caution : We cannot write $\mathbb{R} \setminus 2$ as 2 is not a set !)

Example 1.1.5

Solve $x^2 > 1$.

$\therefore$ $x > 1$ or $x < -1$

OR : $x \in (-\infty, -1) \cup (1, \infty)$

OR : $x \in \mathbb{R} \setminus [-1, 1]$

$\forall$: for all

$\exists$: there exists (at least one)

$\exists!$: there exists unique

$\Rightarrow$: implies

$\Leftrightarrow$: if and only if (equivalent to)

s.t. : such that

Example 1.1.6

$\forall\ y \in (0,\infty),\ \exists\ x \in \mathbb{R}$ s.t. $x^2 = y$

$\downarrow$ translate

For all positive real numbers y, there exists (at least one) real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ or $x = -\sqrt{y}$)

$\forall\ y \in (0,\infty),\ \exists!\ x \in (0,\infty)$ s.t. $x^2 = y$

$\downarrow$ translate

For all positive real numbers y, there exists unique positive real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ only!)

Example 1.1.7

Let $x > 0$,

$$y = \sqrt{x} \overset{\checkmark}{\Rightarrow} y^2 = x$$

$$y^2 = x \overset{\times}{\Rightarrow} y = \sqrt{x} \quad \text{(why?)}$$

Example 1.1.8

In ΔABC,

$\angle ABC = 90^\circ \Rightarrow AB^2 + BC^2 = AC^2$ (Pyth. thm.)

$AB^2 + BC^2 = AC^2 \Rightarrow \angle ABC = 90^\circ$ (Converse of Pyth. thm.)

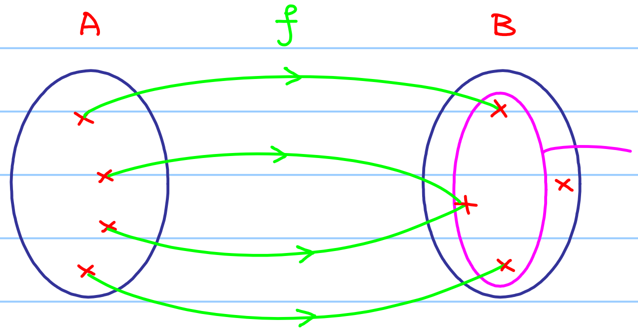
If both statements are true, we say

$\angle ABC = 90^\circ$ if and only if $AB^2 + BC^2 = AC^2$

and we denote it by $\angle ABC = 90^\circ \Leftrightarrow AB^2 + BC^2 = AC^2$

1.2 Functions

Function : A function is a rule that assigns to each element in a set A exactly one element in a set B.



set A : domain (input)

set B : codomain (output)

range(f) $\subseteq$ B : range of f

$\text{range}(f) = f(A) \underset{\substack{\uparrow \\ \text{defined by}}}{:=} \{f(x) \in B : x \in A\}$

A function f from A to B is denoted by $f : A \to B$

Example 1.2.1

If 1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ range(f) = $[0, \infty)$

2) $f : [-1, 2) \to \mathbb{R}$ defined by $f(x) = x^2$ range(f) = $[0, 4)$

Example 1.2.2

If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4$

$f(\underset{\substack{\uparrow \\ \text{input}}}{-3}) = (-3)^2 + 4 = \underset{\substack{\uparrow \\ \text{output}}}{13}$ OR write : $\underset{\substack{\uparrow \\ \text{dependent} \\ \text{variable}}}{y} = \underset{\substack{\uparrow \\ \text{independent} \\ \text{variable}}}{x}{}^2 + 4$

Example 1.2.3

If $f(x) = \frac{2x}{x^2 - 7x}$, find the (maximum) domain of f.

Note : $f(x) = \frac{2x}{x^2-7x}$ is a well-defined function if $x^2 - 7x \neq 0$.

$$x^2 - 7x = 0$$
$$x(x-7) = 0$$
$$x = 0 \text{ or } 7$$

$\therefore$ Domain of $f = \{x \in \mathbb{R} : x \neq 0, 7\}$

$= (-\infty, 0) \cup (0, 7) \cup (7, \infty)$

$= \mathbb{R} \setminus \{0, 7\}$

Example 1.2.4

If $f(x)=\sqrt{x^2-4x+3}$, find the (maximum) domain of f.

Note : $f(x)=\sqrt{x^2-4x+3}$ is a well-defined function if $x^2-4x+3 \geq 0$

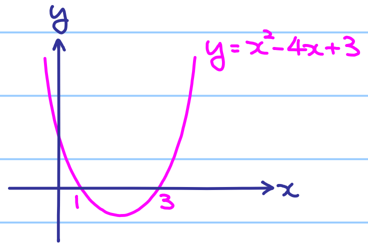
$x^2-4x+3 \geq 0$

$x \leq 1$ or $x \geq 3$

$\therefore$ Domain of $f = \{x \in \mathbb{R} : x \leq 1 \text{ or } x \geq 3\}$

$= (-\infty, 1] \cup [3, \infty)$

$= \mathbb{R} \setminus (1,3)$



Exercise 1.2.1

If $f(x)=\dfrac{1}{\sqrt{x^2-4x+3}}$, find the (maximum) domain of f.

Note : $f(x)=\dfrac{1}{\sqrt{x^2-4x+3}}$ is a well-defined function if $x^2-4x+3>0$.

Ans : Domain of $f = \{x \in \mathbb{R} : x<1 \text{ or } x>3\}$

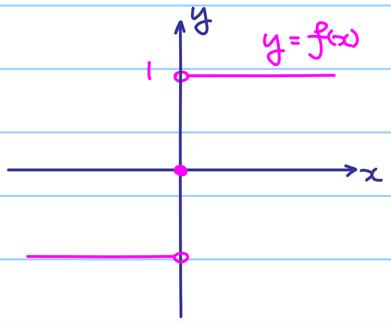
$= (-\infty, 1) \cup (3, \infty)$

$= \mathbb{R} \setminus [1,3]$

Piecewise Defined Function .

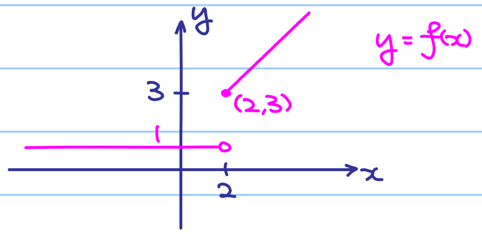
Example 1.2.4

$$\text{If } f(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$$



Example 1.2.5

$$\text{If } f(x) = \begin{cases} x+1 & \text{if } x \geq 2 \\ 1 & \text{if } x<2 \end{cases}$$



Exercise 1.2.2

$$\text{Sketch the graph of } f(x) = \begin{cases} 2x+1 & \text{if } x>1 \\ 0 & \text{if } 0 \leq x \leq 1 \\ -x^2 & \text{if } x<0 \end{cases}$$

Example 1.2.6

Absolute Value : $f(x) = |x| = \sqrt{x^2}$

For example: $|3| = \sqrt{3^2} = \sqrt{9} = 3$

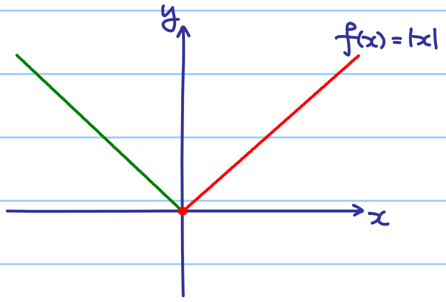
$|0| = \sqrt{0^2} = \sqrt{0} = 0$

$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$

(Simply speaking : throw away the negative sign)

Rewrite $|x|$ as a piecewise defined function :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Example 1.2.7

Let $f(x) = |x+1| + |x-1|$.

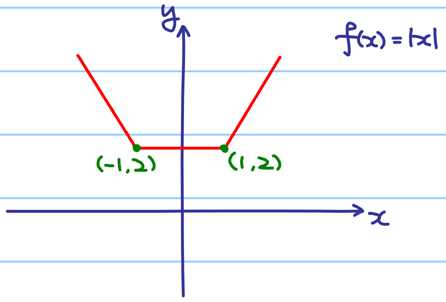
What is the graph of $f(x)$?

Idea: Rewrite $f(x)$ as a piecewise defined function.

Note: $|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \ (\text{i.e } x \geq -1) \\ -(x+1) & \text{if } x+1 < 0 \ (\text{i.e. } x < -1) \end{cases}$

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \ (\text{i.e. } x \geq 1) \\ -(x-1) & \text{if } x-1 < 0 \ (\text{i.e. } x < 1) \end{cases}$$

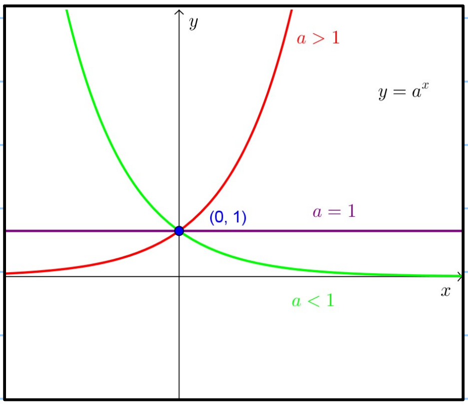
$$\therefore f(x) = |x+1| + |x-1| = \begin{cases} -(x+1) - (x-1) = -2x & \text{if } x < -1 \\ (x+1) - (x-1) = 2 & \text{if } -1 \leq x < 1 \\ (x+1) + (x-1) = 2x & \text{if } x \geq 1 \end{cases}$$



$f(-1) = 2$ and $f(1) = 2(1) = 2$.

Exponential and Logarithmic Functions :

- $y = a^x$ with $a > 0$



graph of $y = a^x$ for

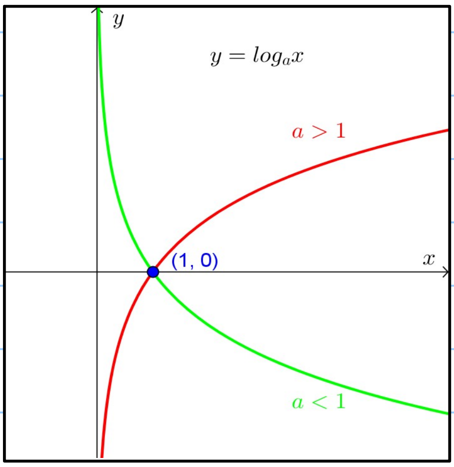
1) $a > 1$ 2) $a = 1$ 3) $0 < a < 1$

Note : $y = a^x$ is well-defined when $a > 0$!

Think: If $a = -1$, when $x = \frac{1}{2}$, $y = a^x = \sqrt{-1}$!

a^x is positive for any $a > 0$ and any real number x .

- $y = \log_a x$ with $a > 1$ or $0 < a < 1$



graph of $y = \log_a x$ for

1) $a > 1$ 2) $0 < a < 1$

Note : $y = \log_a x$ is well-defined when $a > 1$ or $0 < a < 1$!

By definition, if $y = a^x$, then $\log_a y = x$

Facts :

1) $\log_a M + \log_a N = \log_a MN$

2) $\log_a M - \log_a N = \log_a \frac{M}{N}$

3) $\log_a M^n = n \log_a M$

4) $\log_a x = \frac{\log_b x}{\log_b a}$ (Change of base)

5) $e = 2.71828\ldots$ (Explain later)

We write $\log_e x$ as $\ln x$ (natural log function)

6) a^x and $\log_a x$ are inverse to each other,

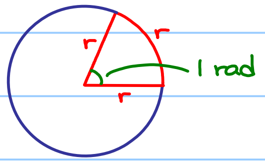
i.e. $a^{\log_a x} = x$ and $\log_a a^x = x$

1.3 Trigonometry

Another unit of measurement of angles (radian):

Definition 1.3.1

When the length of an arc equals to the radius, the angle suspended is defined as 1 radian.



Direct consequence. 2π rad = 360°

Exercise: π rad = _____

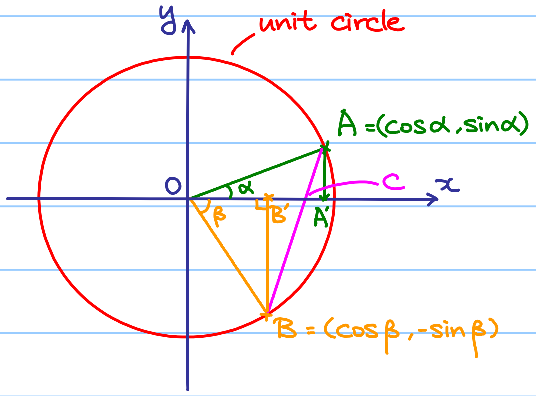
_______ = 90°

_______ = 60°

Remark: From now on, use radian.

Trigonometric identities:

① Consider the length of AB:



i) $AB^2 = OA^2+OB^2-2\cos(\alpha+\beta)$

$= 2-2\cos(\alpha+\beta)$

ii) $AB^2 = (AA'+BB')^2+(A'B')^2$

$=(\sin\alpha+\sin\beta)^2+(\cos\alpha-\cos\beta)^2$

$=2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$

$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$

② Join AB, AB cuts the x-axis at C.

Then $C=\left(\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha+\sin\beta}, 0\right)$

Consider the area of ΔOAB:

i) area of $\Delta OAB = \frac{1}{2}OA\cdot OB\cdot\sin(\alpha+\beta) = \frac{1}{2}\sin(\alpha+\beta)$

ii) area of ΔOAB = area of ΔOAC + area of ΔOBC

$= \frac{1}{2}\cdot OC\cdot AA' + \frac{1}{2}\cdot OC\cdot BB'$

$= \frac{1}{2}\cdot OC\cdot(AA'+BB')$

$= \frac{1}{2}\cdot\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha+\sin\beta}\cdot(\sin\alpha+\sin\beta)$

$= \frac{1}{2}\cdot(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$

$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$

Theorem 1.3.1

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Compound angle formula

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\Big\downarrow$ replace β by $-\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\Big\downarrow$ replace β by $-\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\downarrow$ taking quotient of the 1st and the 3rd eq^n

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double angle formula

$$\xrightarrow{\text{put } \beta=\alpha} \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\xrightarrow{\text{put } \beta=\alpha} \sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\xrightarrow{\text{put } \beta=\alpha} \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Product to sum formula

$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

put $\alpha = \frac{A+B}{2}$, $\beta = \frac{A-B}{2}$

Sum to product formula

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

1.4 Parametrized Curves

$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$ = set of points of the coordinate plane

A parametrized curve in $\mathbb{R}^2$ is a function $\gamma : I \to \mathbb{R}^2$, where I is an interval.

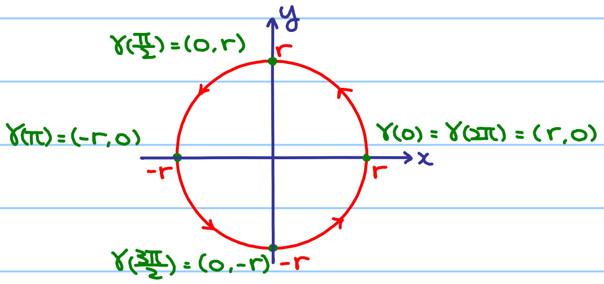
We can also write $\gamma(t) = (x(t), y(t))$, where $t \in I$.

Idea: Regard the variable t as time, $\gamma(t) = (x(t), y(t))$ gives the position of a moving particle on xy-plane for a given time t, and the curve is the locus of the particle.

Example 1.4.1

Let $\gamma(t) = (x(t), y(t)) = (r\cos t, r\sin t)$, for $t \in [0, 2\pi]$, $r > 0$.

$$\begin{cases} x = r\cos t & -(1) \\ y = r\sin t & -(2) \end{cases}$$



Eliminate t:

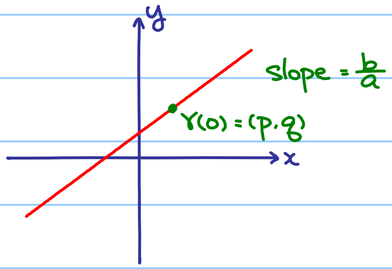
$(1)^2 + (2)^2 : \; x^2 + y^2 = r^2$

so γ defines a circle centered at the origin with radius r.

Example 1.4.2

Let $\gamma(t) = (x(t), y(t)) = (p + ta, q + tb)$, for $t \in \mathbb{R}$, $a, b \neq 0$.

$$\begin{cases} x = p + ta & -(1) \\ y = q + tb & -(2) \end{cases}$$



Eliminate t:

$\frac{x-p}{a} = \frac{y-q}{b} \; (= t)$

$\frac{y-q}{x-p} = \frac{b}{a}$

so γ defines the straight line passing through (p,q) with slope $\frac{b}{a}$.

1.5 Sequences of Real Numbers

Example 1.5.1

Let $a_1 = 2$, $a_2 = \pi$, $a_3 = \sqrt{3}$, $\cdots$

OR write as $\{2, \pi, \sqrt{3}, \cdots\}$ (No pattern)

Example 1.5.2

Sequences having patterns :

Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $\cdots$ in general , $a_n = 2^{n-1}$

Let $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $\cdots$ in general , $a_n = \frac{1}{n}$

Let $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $\cdots$ in general , $a_n = (-1)^n$

Example 1.5.3

Recursive sequence.

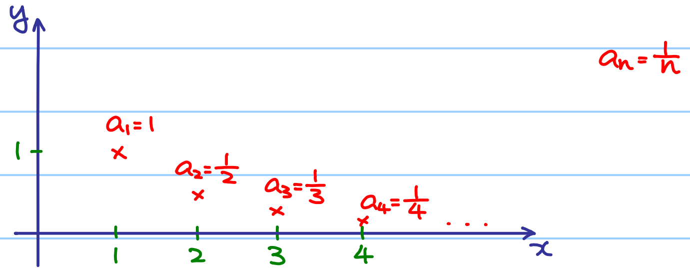
Let $\{a_n\}$ be a sequence of real numbers defined by $a_1 = 1$ and $a_{n+1} = a_n^2 + 2$ for $n \geq 1$.

Then $\{a_n\} = \{1, 3, 11, 123, \cdots\}$.

Remark / Definition 1.5.1

A sequence of real numbers $\{a_n\}$ can be regarded as a function $f : \mathbb{Z}^+ \to \mathbb{R}$ and $a_n = f(n)$ (i.e. given $n \in \mathbb{Z}^+$, return the n-th term of the sequence.)

A sequence can be visualized by the following diagram :



Any observation ?

When n is getting larger and larger , a_n is getting closer and closer to 0 .

§2 Limits of Sequences

2.1 Definition

Definition 2.1.1 (Informal)

Let $\{a_n\}$ be a sequence of real numbers.

If n is getting larger and larger, a_n is getting closer and closer to $L \in \mathbb{R}$, then we say L is the limit of the sequence $\{a_n\}$ and we denote it by $\lim\limits_{n\to\infty} a_n = L$.

In this case, $\{a_n\}$ is said to be convergent.

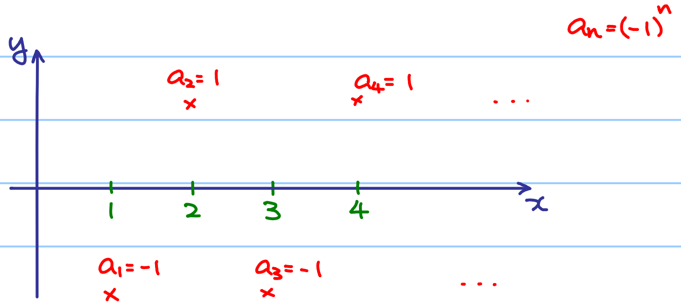
Example 2.1.1

$\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

$\lim\limits_{n\to\infty} 2^{n-1}$ does NOT exist.

(But some still write $\lim\limits_{n\to\infty} 2^{n-1} = +\infty$ or say 2^{n-1} diverges to $+\infty$)

$\lim\limits_{n\to\infty} (-1)^n$ does NOT exist.



Theorem 2.1.1

1) If $a_n = k$ for all $n \in \mathbb{Z}^+$ (constant sequence), then $\lim\limits_{n\to\infty} a_n = k$.

2) If $k > 0$ and $a_n = n^{-k} = \frac{1}{n^k}$ for all $n \in \mathbb{Z}^+$, then $\lim\limits_{n\to\infty} a_n = 0$.

3) If $-1 < a < 1$, then $\lim\limits_{n\to\infty} a^n = 0$.

2.2 Algebraic Properties of Limits

Theorem 2.2.1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ (very important assumption), then

1) $\lim\limits_{n\to\infty} a_n + b_n = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L + M$

2) $\lim\limits_{n\to\infty} a_n - b_n = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = L - M$

3) $\lim\limits_{n\to\infty} a_n b_n = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n) = LM$

4) If $M \neq 0$, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L}{M}$

Example 2.2.1

Find $\lim_{n\to\infty} \frac{2}{n} + 3$

Logically :

① $\lim_{n\to\infty} 2 = 2$, $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} \frac{2}{n} \overset{\overset{\text{By (3)}}{\downarrow}}{=} (\lim_{n\to\infty} 2)(\lim_{n\to\infty} \frac{1}{n}) = 2 \cdot 0 = 0$

② $\lim_{n\to\infty} \frac{2}{n} = 0$, $\lim_{n\to\infty} 3 = 3$, so $\lim_{n\to\infty} \frac{2}{n} + 3 \overset{\overset{\text{By (1)}}{\downarrow}}{=} \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3 = 0 + 3 = 3$

But what we write :

$$\lim_{n\to\infty} \frac{2}{n} + 3 = \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3$$

$$= 0 + 3$$

$$= 3$$

Example 2.2.2

Find $\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$

$\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$ (We cannot use (4) , why ?)

$= \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{2-\frac{4}{n}}$ (Now, we can use (4) !)

$= \frac{\lim_{n\to\infty} 1+\frac{3}{n^2}}{\lim_{n\to\infty} 2-\frac{4}{n}}$

$= \frac{1}{2}$

Exercise 2.2.1

Find $\lim_{n\to\infty} \frac{3n+1}{n^2-2n}$, $\lim_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ (if exist)

Answer : $\lim_{n\to\infty} \frac{3n+1}{n^2-2n \leftarrow \text{grows faster}}$ $\lim_{n\to\infty} \frac{n^3+2n \leftarrow \text{grows faster}}{2n^2+1}$

$\lim_{n\to\infty} \frac{3n+1}{n^2-2n} = 0$, $\lim_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ does NOT exist

Any observation ?

Basically, we are comparing the degrees of the numerator and the denominator.

Conclusion :

If $p(x)$ and $q(x)$ are polynomials ,

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \neq 0$ ($\deg p(x) = m$)

$q(x) = b_k x^k + a_{k-1} x^{k-1} + \cdots + b_1 x + b_0$ with $b_k \neq 0$ ($\deg q(x) = k$)

then

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \pm\infty & \text{if } m > k \\ \frac{a_m}{b_k} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Following this idea :

Example 2.2.3

Find $\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$

$$\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}} \longleftarrow \text{roughly } \deg = 1$$

$$= \lim_{n\to\infty} \frac{3-\frac{1}{n}}{\sqrt{4+\frac{2}{n}}}$$

$$= \frac{3}{2}$$

Example 2.2.4

Find $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$ (Never say $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \infty - \infty = 0$)

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$$

$$= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= 0$$

Example 2.2.5

Find $\lim_{n\to\infty} \frac{2^n}{n}$.

Question : Can we say $\frac{2^n}{n} = \frac{1}{n} \cdot 2^n$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} \frac{2^n}{n} = 0$?

Absolutely NOT !

Since $\lim_{n\to\infty} 2^n$ does NOT exist , property (3) cannot be applied !

2.3 Constant e

Consider a number $(1+\frac{1}{m})^n$ which depends on both m and n and then

1) fix m, say m = 100, n is getting larger and larger.

$n = 10$	$n = 100$	$n = 1000$	$n \to \infty$
$(1+\frac{1}{m})^n = 1.01^{10}$	$(1+\frac{1}{m})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.01^{1000}$	$(1+\frac{1}{m})^n \to \infty$

2) fix n, say n = 100, m is getting larger and larger.

$m = 10$	$m = 100$	$m = 1000$	$m \to \infty$
$(1+\frac{1}{m})^n = 1.1^{100}$	$(1+\frac{1}{m})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.001^{100}$	$(1+\frac{1}{m})^n \to 1$

How about setting $m = n$ and let them become larger and larger ?

$\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists ? something between 1 and ∞

$n = 10$	$n = 100$	$n = 1000$	$n \to \infty$
$(1+\frac{1}{n})^n = 1.1^{10}$	$(1+\frac{1}{n})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.001^{1000}$	$(1+\frac{1}{n})^n \to 2.71828\ldots$
≈ 2.59374	≈ 2.70481	≈ 2.71692	limit exists and call it e.

Theorem 2.3.1

$\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists and the limit is called e

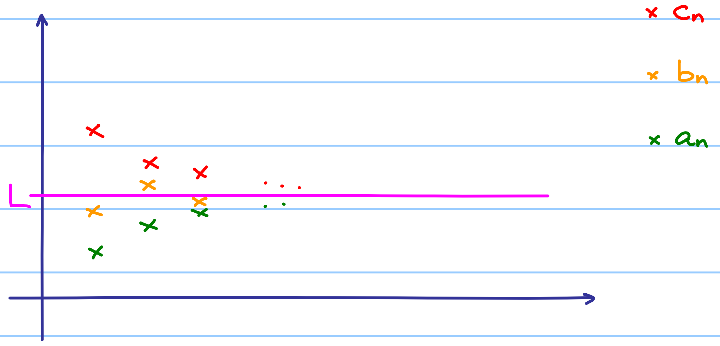
2.4 Sandwich Theorem

Theorem 2.4.1 (Sandwich Theorem)

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers.

If $a_n \le b_n \le c_n$ for all $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Geometrical meaning :



In fact, the result is still true if $a_n \le b_n \le c_n$ for all $n \ge n_0$.

Idea : Estimate a sequence $\{b_n\}$ that we do not understand very well by sequences $\{a_n\}$ and $\{c_n\}$ that we understand well.

Example 2.4.1

Find $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Note : $0 \leqslant \frac{1}{\sqrt{n+1}+\sqrt{n}} \leqslant \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{Z}^+$ and $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$

By sandwich theorem , $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

Example 2.4.2

Find $\lim\limits_{n\to\infty} \frac{1}{n}\sin n$

Note : $-\frac{1}{n} \leqslant \frac{1}{n}\sin n \leqslant \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ and $\lim\limits_{n\to\infty} -\frac{1}{n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

By sandwich theorem , $\lim\limits_{n\to\infty} \frac{1}{n}\sin n = 0$.

Exercise 2.4.1

Prove that $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2} = 0$

Hint : $-1 \leq (-1)^n \leq 1$

Exercise 2.4.2

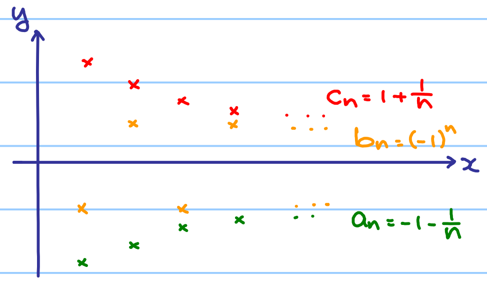
If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{Z}^+$ and $\lim\limits_{n\to\infty} a_n = -1$, $\lim\limits_{n\to\infty} c_n = 1$,

can we conclude that $-1 \leq \lim\limits_{n\to\infty} b_n \leq 1$?

No! Consider $a_n = -1-\frac{1}{n}$, $b_n = (-1)^n$, $c_n = 1+\frac{1}{n}$ for all $n \in \mathbb{Z}^+$

We have $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{Z}^+$ and $\lim\limits_{n\to\infty} a_n = -1$, $\lim\limits_{n\to\infty} c_n = 1$,

however $\lim\limits_{n\to\infty} b_n$ does not exist .



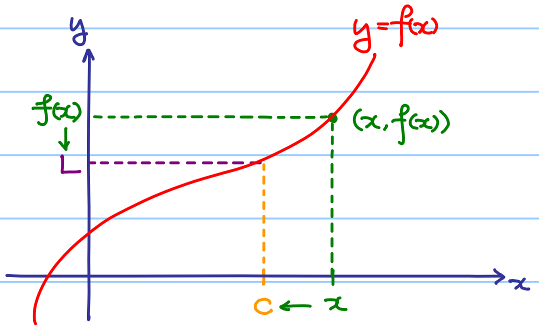
§3 Limits of Functions

3.1 Definition

Definition 3.1.1 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer^{+} to c from both sides, then L is called the limit of $f(x)$ at c, and we write $\lim_{x \to c} f(x) = L$.

In this case, $f(x)$ is said to be convergent to L as x tends to c.

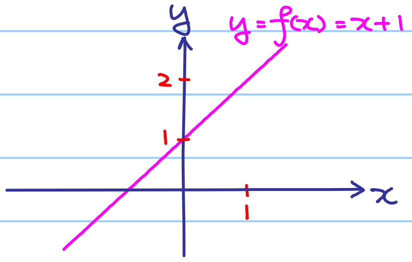


+ Note : A little bit misleading !

$f(c)$ may NOT equal to L, even it may be undefined !

Example 3.1.1

If $f(x) = x+1$, find $\lim_{x \to 1} f(x)$.



+

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1 9	1 99	1 999	2	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

We write $\lim_{x \to 1} f(x) = 2$.

Remarks :

1) + The table only gives an intuitive idea, but NOT a rigorous proof !

2) Always Remember :

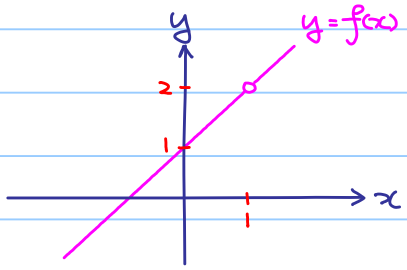
Do NOT regard finding limit as putting $x=1$ into $f(x)$ and getting $f(1)=2$!

Example 3.1.2

Let $f(x)$ be a function defined by $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

We can rewrite f as the following:

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

(But, we do NoT care what happens when $x=1$!)

We still have $\lim_{x \to 1} f(x) = 2$.

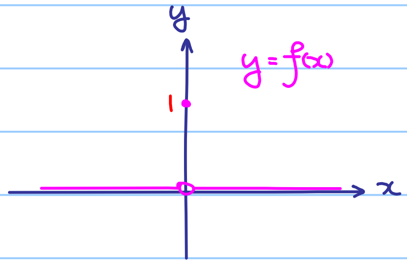
Compare with the previous example !

Idea: When x is "near" 1, both x^2-1 and $x-1$ are small,

but the quotient of them is not small !

Example 3.1.3

Let $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



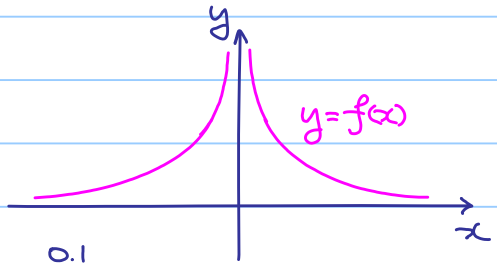
x	-0.1	-0.01	-0.001	$\boxed{0}$	0.001	0.01	0.1
$f(x)$	0	0	0	$\boxed{1}$	0	0	0

Do NoT care !

$\lim_{x \to 0} f(x) = 0$ which does NoT equal to $f(0) = 1$.

Example 3.1.4

Let $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$.



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	undefined	10^6	10^4	10^2

$f(x)$ tends to $+\infty$ (NOT a real number) as x tends to 0.

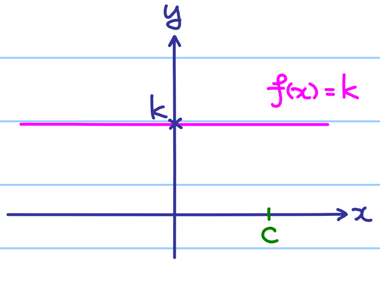
$\therefore \lim\limits_{x \to 0} f(x)$ does NOT exist.

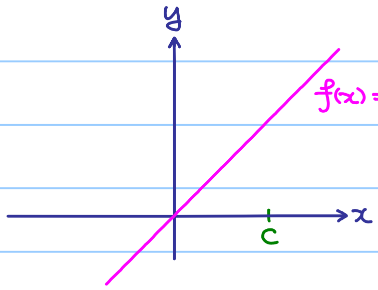
(But some still write $\lim\limits_{x \to 0} f(x) = +\infty$ or say $f(x)$ diverges to $+\infty$ as x tends to 0.)

Theorem 3.1.1

1) If k is a constant, then $\lim\limits_{x \to c} k = k$ regarded as a constant function $f(x) = k$.

2) $\lim\limits_{x \to c} x = c$.





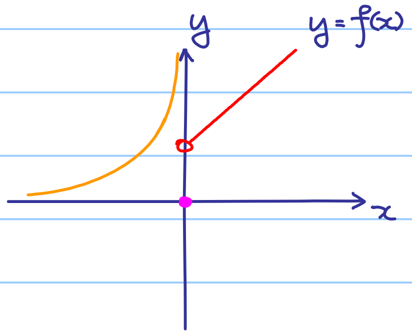
Definition 3.1.3 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer to c from the right (left) hand side, then L is called the right (left) hand limit of $f(x)$ at c.

We denote it by $\lim\limits_{x \to c^+} f(x) = L$ ($\lim\limits_{x \to c^-} f(x) = L$).

Example 3.1.5

$$\text{If } f(x) = \begin{cases} x+1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ \frac{1}{x^2} & \text{if } x<0 \end{cases}$$



$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+1 = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2} \quad (\text{does NOT exist})$$

$$f(0) = 0$$

Remark:

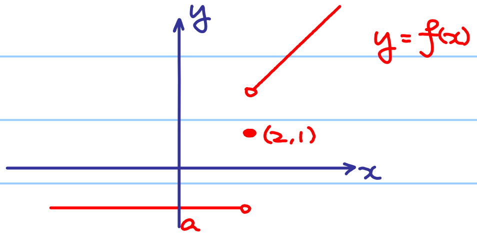
Right hand limit and left hand limit of a function at a point are NOT necessary to be the same!

Theorem 3.1.2

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

Example 3.1.6

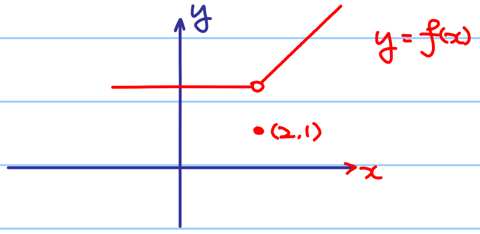
$$\text{If } f(x) = \begin{cases} x & \text{if } x \geqslant 2 \\ 1 & \text{if } x=2 \\ a & \text{if } x<2 \end{cases}$$



Given that $\lim_{x\to 2} f(x)$ exists. What is the value of a?

$\lim_{x\to 2} f(x)$ exists $\Rightarrow$ Both $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ exist

and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$



$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$

$$\lim_{x\to 2^+} x = \lim_{x\to 2^-} a$$

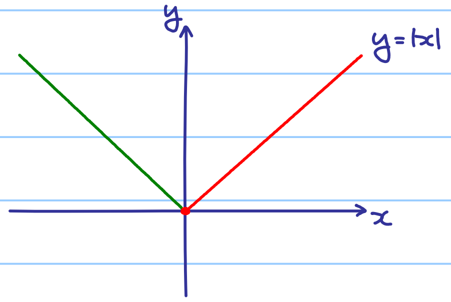
$$2 = a$$

$\lim_{x\to 2} f(x)$ exists, it forces $a=2$!

Remark: It is nothing related to $f(2)=1$.

Example 3.1.7

Let $f(x)=|x|$ $\qquad f(x)=\begin{cases} x & \text{if} \quad x\geqslant 0 \\ -x & \text{if} \quad x<0 \end{cases}$



$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x = 0$

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} -x = 0$

Don't skip!

$\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) = 0$ and so $\lim_{x\to0} |x| = 0$.

Remark : We cannot say $\lim_{x\to0} f(x) = \lim_{x\to0} x$ or $\lim_{x\to0} f(x) = \lim_{x\to0} -x$

since when we find $\lim_{x\to0} f(x)$, we need to consider the neighborhood of 0.

However, $\lim_{x\to2} f(x) = \lim_{x\to2} x = 2$ since $f(x)=x$ in an neighborhood of 2.

3.2 Algebraic Properties of Limits

Theorem 3.2.1

If both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist (Very important assumption!), then

(1) $\lim_{x\to c} f(x)+g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(2) $\lim_{x\to c} f(x)-g(x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

(3) $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \ \lim_{x\to c} g(x)$

(4) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$

Example 3.2.1

Find $\lim_{x\to2} 3x^2-5$.

Logically :

① $\lim_{x\to2} x = 2$, so $\lim_{x\to2} x^2 = \lim_{x\to2} (x\cdot x) \overset{(3)}{=} \lim_{x\to2} x \cdot \lim_{x\to2} x = 2\cdot2 = 4$

② $\lim_{x\to2} 3 = 3$, $\lim_{x\to2} x^2 = 4$, so $\lim_{x\to2} 3x^2 \overset{(3)}{=} \lim_{x\to2} 3 \cdot \lim_{x\to2} x^2 = 3\cdot4 = 12$

③ $\lim_{x\to2} 3x^2 = 12$, $\lim_{x\to2} 5 = 5$, so $\lim_{x\to2} 3x^2-5 \overset{(2)}{=} \lim_{x\to2} 3x^2 - \lim_{x\to2} 5 = 12-5 = 7$

But what we write :

$$\begin{aligned}\lim_{x\to2} 3x^2-5 &= 3(\lim_{x\to2} x)^2 - 5 \\ &= 3\cdot2^2-5 \\ &= 7\end{aligned}$$

Example 3.2.2

Find $\lim\limits_{x\to1}\dfrac{3x^2-8}{x-2}$

$$\lim_{x\to1}\frac{3x^2-8}{x-2}=\frac{\lim\limits_{x\to1}3x^2-8}{\lim\limits_{x\to1}\ x-2}=\frac{3(\lim\limits_{x\to1}x)^2-8}{(\lim\limits_{x\to1}x)-2}=\frac{3\cdot1^2-8}{1-2}=5$$

Caution!

It seems that it makes no difference by putting $x=1$, and then

$$\lim_{x\to1}\frac{3x^2-8}{x-2}=\frac{3\cdot1^2-8}{1-2}=5$$

But, think carefully! Let $f(x)=\dfrac{3x^2-8}{x-2}$, how do you know $\lim\limits_{x\to1}f(x)=f(1)$?

Things will become clear when we discuss continuity of functions!

Example 3.2.3

Find $\lim\limits_{x\to1}\dfrac{x^2-1}{x^2-3x+2}$

Note: $\lim\limits_{x\to1}x^2-3x+2=0$, so we cannot use (4).

$$\lim_{x\to1}\frac{x^2-1}{x^2-3x+2}=\lim_{x\to1}\frac{(x-1)(x+1)}{(x-1)(x-2)}=\lim_{x\to1}\frac{x+1}{x-2}\overset{(4)}{=}\frac{\lim\limits_{x\to1}x+1}{\lim\limits_{x\to1}x-2}=\frac{2}{-1}=-2$$

$\because x\neq1$

$\therefore x-1\neq0$ and division can be done!

Example 3.2.4

Let $f:\mathbb{R}\setminus\{1\}\longrightarrow\mathbb{R}$ defined by $f(x)=\dfrac{\sqrt{x}-1}{x-1}$.

Find $\lim\limits_{x\to1}f(x)$.

$$\lim_{x\to1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to1}\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$ (Something like rationalization)

$$=\lim_{x\to1}\frac{x-1}{(x-1)(\sqrt{x}+1)}$$

$$=\lim_{x\to1}\frac{1}{\sqrt{x}+1}$$

$$=\frac{1}{2}$$

Example 3.2.5

$$\lim_{x\to0}\frac{1}{x}=\lim_{x\to0}x\cdot\frac{1}{x^2}\overset{(*)}{=}\lim_{x\to0}x\cdot\lim_{x\to0}\frac{1}{x^2}=0\cdot\lim_{x\to0}\frac{1}{x^2}=0$$ Anything wrong?

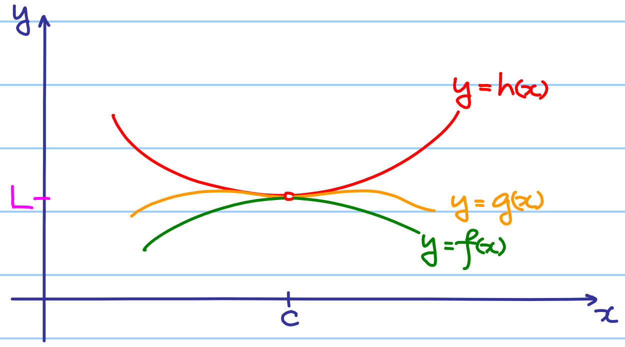
$\lim\limits_{x\to0}\dfrac{1}{x^2}$ does NOT exist, so we cannot use (3) at (*).

3.3 Sandwich Theorem for Functions

Theorem 3.3.1

If $f(x) \le g(x) \le h(x)$ for all $x \in \mathbb{R} \setminus \{c\}$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Geometrical meaning:



In fact, the result is still true if $f(x) \le g(x) \le h(x)$ holds in an open interval containing c but possibly except c.

Example 3.3.1

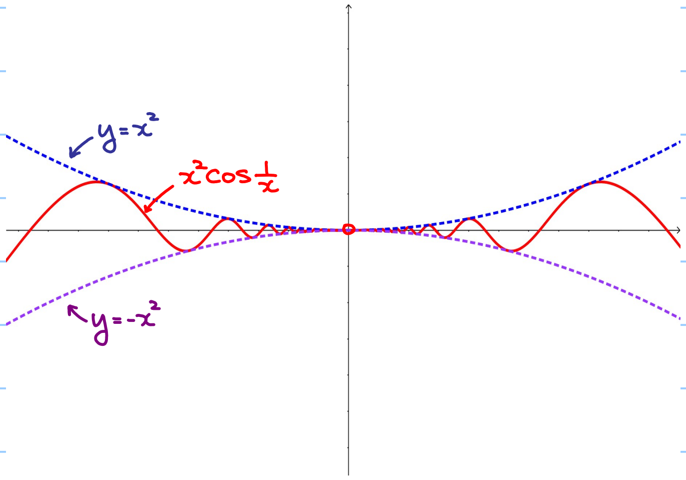
Prove that $\lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$

Note that: $-1 \le \cos\frac{1}{x} \le 1$ for $x \neq 0$

$$-x^2 \le x^2 \cos\frac{1}{x} \le x^2$$

and $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$

By sandwich theorem, $\lim_{x\to 0} x^2 \cos^2 \frac{1}{x} = 0$.



Remark:

Sandwich theorem can be generalized to left and right hand limit.

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions and $c \in \mathbb{R}$

If $f(x) \le g(x) \le h(x)$ for all $x < c$ $(x > c)$ and $\lim_{x\to c^-} f(x) = \lim_{x\to c^-} h(x) = L$ $\left(\lim_{x\to c^+} f(x) = \lim_{x\to c^+} h(x) = L\right)$

then $\lim_{x\to c^-} g(x) = L$ $\left(\lim_{x\to c^+} g(x) = L\right)$.

Exercise 3.3.1

Prove that $\lim_{x\to 1^+} (x^2-1)\sin\left(\frac{1}{\sqrt{x-1}}\right) = 0$

Example 3.3.2

Prove that $\lim_{x\to 0} x\cos\frac{1}{x} = 0$

Note that $-1 \leq \cos\frac{1}{x} \leq 1$ for $x \neq 0$

$-|x| \leq x \leq |x|$

$\therefore\ -|x| \leq x\cos\frac{1}{x} \leq |x|$ for $x \neq 0$

Also $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$,

by the sandwich theorem, $\lim_{x\to 0} x\cos\frac{1}{x} = 0$

Theorem 3.3.2

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Idea: When x becomes small (but not zero), both $\sin x$ and x are small,

but the quotient of them is not small!

proof:

1) Consider $0 < x < \frac{\pi}{2}$, we have

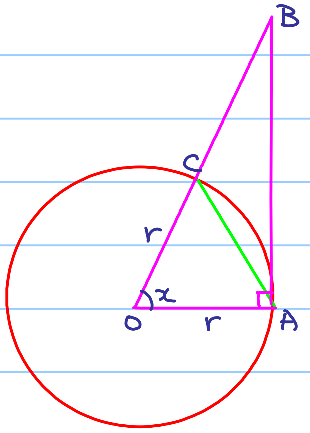
Area of ΔOAC < Area of sector OAC < Area of ΔOAB

$\frac{1}{2}r^2\sin x < \frac{1}{2}r^2 x < \frac{1}{2}r^2\tan x$

$\sin x < x < \tan x$

⇓ ⇓

$\frac{\sin x}{x} < 1$ $\cos x < \frac{\sin x}{x}$



$\therefore\ \cos x < \frac{\sin x}{x} < 1$ for $0 < x < \frac{\pi}{2}$,

Also, $\lim_{x\to 0^+} \cos x = \lim_{x\to 0^+} 1 = 1$, therefore by the sandwich theorem, $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$.

2) Consider $-\frac{\pi}{2} < x < 0$, we have

Let $y = -x$, then $0 < y < \frac{\pi}{2}$, so

$\cos y < \frac{\sin y}{y} < 1$

$\cos(-x) < \frac{\sin(-x)}{-x} < 1$

$\therefore\ \cos x < \frac{\sin x}{x} < 1$ for $-\frac{\pi}{2} < x < 0$.

Also, $\lim_{x\to 0^-} \cos x = \lim_{x\to 0^-} 1 = 1$, therefore by the sandwich theorem, $\lim_{x\to 0^-} \frac{\sin x}{x} = 1$.

$\therefore$ By (1) and (2), $\lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\sin x}{x} = 1$, therefore $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Example 3.3.3

Find $\lim_{x \to 0} \frac{\sin 3x}{2x}$.

$$\lim_{x \to 0} \frac{\sin 3x}{2x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{2} = 1 \cdot \frac{3}{2} = \frac{3}{2}$$

Example 3.3.4

Find $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$

$$\begin{aligned}\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2} &= \lim_{x \to 0} \frac{2 \sin \frac{a+b}{2}x \sin \frac{b-a}{2}x}{x^2} \\ &= \lim_{x \to 0} 2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) \frac{\sin \frac{a+b}{2}x}{\frac{a+b}{2}x} \frac{\sin \frac{b-a}{2}x}{\frac{b-a}{2}x} \\ &= \frac{b^2-a^2}{2}\end{aligned}$$

3.4 Limits at Infinity

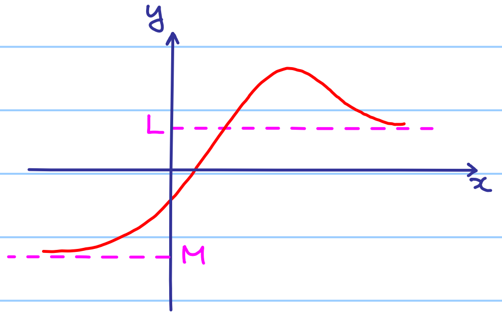
Definition 3.4.1 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets bigger and bigger (as x goes to $+\infty$), then L is called the limit of $f(x)$ at $+\infty$. We write $\lim_{x \to +\infty} f(x) = L$.

(Similar definition for $\lim_{x \to -\infty} f(x)$)

From the graph, we have

$\lim_{x \to +\infty} f(x) = L$ but $\lim_{x \to -\infty} f(x) = M$.



$\therefore$ $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ are NOT necessary to be the same!

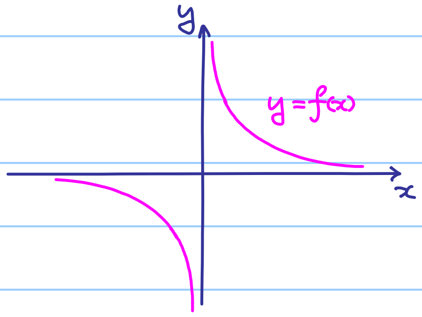
However if $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = L$, some simply write $\lim_{x \to \pm\infty} f(x) = L$.

Example 3.4.1

Let $f(x) = \frac{1}{x}$, $x \neq 0$.

Then $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0$,

or simply write $\lim_{x \to \pm\infty} f(x) = 0$.



Theorem 3.4.1

1) If $k > 0$, then $\lim_{x \to +\infty} \frac{1}{x^k} = 0$.

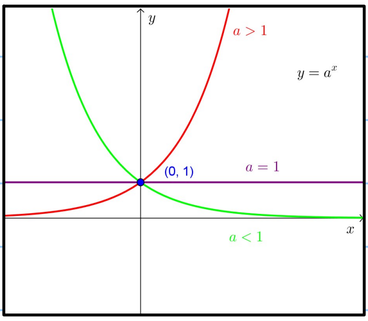
2) $\lim_{x \to \pm\infty} (1 + \frac{1}{x})^x = e \approx 2.71828$

Theorem 3.4.2

If $a>1$, $\lim\limits_{x\to-\infty} a^x = 0$

If $1>a>0$, $\lim\limits_{x\to+\infty} a^x = 0$

$\lim\limits_{x\to\pm\infty} 1^x = 1$



3.5 Algebraic Properties of Limits at Infinity

Theorem 3.5.1

If both $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to+\infty} g(x)$ exist (Very important assumption!), then

(1) $\lim\limits_{x\to+\infty} f(x)+g(x) = \lim\limits_{x\to+\infty} f(x) + \lim\limits_{x\to+\infty} g(x)$

(2) $\lim\limits_{x\to+\infty} f(x)-g(x) = \lim\limits_{x\to+\infty} f(x) - \lim\limits_{x\to+\infty} g(x)$

(3) $\lim\limits_{x\to+\infty} f(x)g(x) = \lim\limits_{x\to+\infty} f(x) \cdot \lim\limits_{x\to+\infty} g(x)$

(4) $\lim\limits_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to+\infty} f(x)}{\lim\limits_{x\to+\infty} g(x)}$ if $\lim\limits_{x\to+\infty} g(x) \neq 0$.

Similar results hold for limits at $-\infty$

Example 3.5.1

Find $\lim\limits_{x\to+\infty} \frac{3x^2}{x^2+x+1}$

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} \;\; \neq \;\; \frac{\lim\limits_{x\to+\infty} 3x^2}{\lim\limits_{x\to+\infty} x^2+x+1}$$

Both limits does NOT exist.

$$= \lim_{x\to+\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}}$$

$$= \lim_{x\to+\infty} \frac{3}{1+0+0}$$

$$= 3$$

Example 3.5.2

Find $\lim\limits_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$

$$\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$$

$$= \lim_{x\to+\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{3-\frac{2}{x}+\frac{1}{x^2}}$$

$$= \frac{0+0}{3+0+0}$$

$$= 0$$

Conclusion :

If $p(x)$ and $q(x)$ are polynomials

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \neq 0$ (i.e. deg $p(x) = m$)

$q(x) = b_n x^n + a_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ with $b_n \neq 0$ (i.e. deg $q(x) = n$)

then

$$\lim_{x \to +\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty / -\infty & \text{if } m > k \\ \frac{a_m}{b_m} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Similar result as the case in limits of sequences !

Example 3.5.3

Find $\lim_{x \to -\infty} \frac{x}{\sqrt{4x^2+1}}$

$$\lim_{x \to -\infty} \frac{x}{\sqrt{4x^2+1}}$$

← deg 1

← roughly, deg 1

⇒ limit should exist !

$$= \lim_{x \to -\infty} \frac{1}{\frac{1}{x}\sqrt{4x^2+1}}$$

$$= \lim_{x \to -\infty} \frac{1}{-\sqrt{\frac{1}{x^2}} \cdot \sqrt{4x^2+1}}$$

(Caution : $x < 0 \Rightarrow \frac{1}{x} = -\sqrt{\left(\frac{1}{x}\right)^2} = -\sqrt{\frac{1}{x^2}}$)

$$= \lim_{x \to -\infty} -\frac{1}{\sqrt{4 + \frac{1}{x^2}}}$$

$$= -\frac{1}{2}$$

Following this idea, we are going to compare exponential functions and polynomials .

Theorem 3.5.2

1) $\lim_{x \to +\infty} x^k e^{-x} = \lim_{x \to +\infty} \frac{x^k}{e^x} = 0$, for any $k > 0$.

2) $\lim_{x \to +\infty} p(x) e^{-x} = \lim_{x \to +\infty} \frac{p(x)}{e^x} = 0$, for any polynomial $p(x)$.

Roughly speaking : As $x \to +\infty$, e^x grows "faster" than any polynomial .

Proof can be done when L'Hôptial's rule is covered.

Example 3.5.4

Find $\lim\limits_{x\to+\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$ and $\lim\limits_{x\to-\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

$$\lim_{x\to+\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$$ dominating terms

$$=\lim_{x\to+\infty}\frac{1+e^{-2x}}{1-e^{-2x}}$$
$$=\frac{1+0}{1-0}$$
$$=1$$

$$\lim_{x\to-\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$$ dominating terms

$$=\lim_{x\to-\infty}\frac{e^{2x}+1}{e^{2x}-1}$$
$$=\frac{0+1}{0-1}$$
$$=-1$$

💡 Idea: Taking quotient of the dominating term.

3.6 Limits Involving e

Example 3.6.1

Find $\lim\limits_{x\to+\infty}(1+\dfrac{1}{2x-1})^x$

$$\lim_{x\to+\infty}(1+\frac{1}{2x-1})^x=\lim_{x\to+\infty}(1+\frac{1}{2x-1})^{\frac{1}{2}(2x-1)+\frac{1}{2}}$$
$$=\lim_{x\to+\infty}[(1+\frac{1}{2x-1})^{2x-1}]^{\frac{1}{2}}\cdot(1+\frac{1}{2x-1})^{\frac{1}{2}}$$
$$=e^{\frac{1}{2}}\cdot 1$$
$$=e^{\frac{1}{2}}$$

Example 3.6.2

Find $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$.

Let $y=\frac{1}{x}$, as $x\to 0$, $y\to\pm\infty$ (Not only $+\infty$, but also $-\infty$)

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{y\to\pm\infty}(1+\frac{1}{y})^y=e$$

Next, consider $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}$.

💡 Idea: When x becomes small (but not zero), both e^x-1 and x are small,

but the quotient of them is not small!

Theorem 3.6.1

$\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1.$

Cheating : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = \lim_{x\to 0} \quad 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \overset{(*)}{=} 1$$

(*) is cheating since we are summing up infinitely many small terms , so algebraic properties of limits (theorem 3.2.1) cannot be applied .

Exercise 3.6.1

Use a calculator and fill the following table to convince yourself that $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{e^x-1}{x}$				undefined			

Example 3.6.2

Find $\lim\limits_{x\to 0} \frac{e^{3x}-1}{2x}$.

$$\lim_{x\to 0} \frac{e^{3x}-1}{2x} = \lim_{x\to 0} \frac{e^{3x}-1}{3x} \cdot \frac{3}{2}$$

$$= \left(\lim_{x\to 0} \frac{e^{3x}-1}{3x}\right) \cdot \left(\lim_{x\to 0} \frac{3}{2}\right)$$

$$= 1 \cdot \frac{3}{2}$$

$$= \frac{3}{2}$$

3.7 Sandwich Theorem at Infinity

Theorem 3.7.1

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions.

If $f(x) \le g(x) \le h(x)$ for all $x \in \mathbb{R}$

and $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} h(x) = L$, then $\lim_{x \to +\infty} g(x) = L$.

Geometrical meaning :

In fact, the result is still true if $f(x) \le g(x) \le h(x)$ for all $x \in [a, +\infty)$

Similar result holds for limits at $-\infty$

Example 3.7.1

Find $\lim_{x \to +\infty} e^{-x} \sin x$

Since $-1 \le \sin x \le 1$ and $e^{-x} > 0$

$$-e^{-x} \le e^{-x} \sin x \le e^{-x}$$

Note : $\lim_{x \to +\infty} -e^{-x} = \lim_{x \to +\infty} e^{-x} = 0$.

By the sandwich theorem, $\lim_{x \to +\infty} e^{-x} \sin x = 0$.

Exercise 3.7.1

Show that $\lim_{x \to +\infty} \frac{\sin x}{x} = 0$.

(Don't mix up with $\lim_{x \to 0} \frac{\sin x}{x} = 1$)

§ 4 Continuity

4.1 Definition

Definition 4.1.1

A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

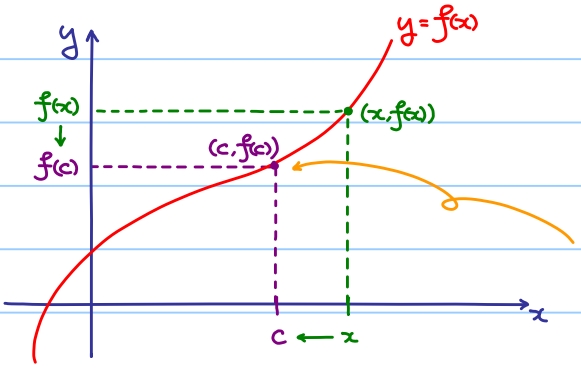
Idea:

③ they equal

$$\lim_{x \to c} f(x) = f(c)$$

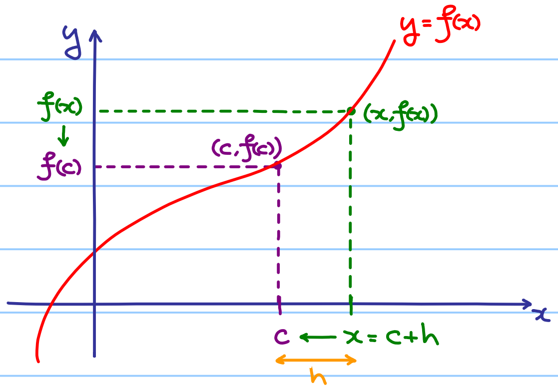
① limit exists   ② well-defined

Geometrical meaning:



the curve does NOT break up at the point $x = c$!

Furthermore, if a function f is continuous at every point whenever it is defined, then f is said to be a continuous function.



Let $h = x - c$, i.e. $x = c + h$ (Remark: When $x < c$, we have $h < 0$.)

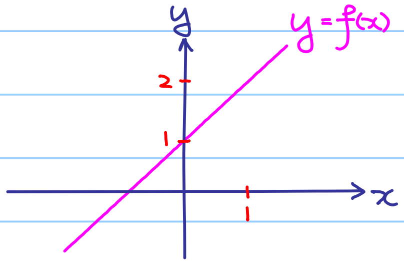
When x tends to c, h tends to 0.

Therefore, we have another formulation:

A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{h \to 0} f(c+h) = f(c)$.

Example 4.1.1

Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x+1$.



We have : ① $\lim\limits_{x\to1} f(x) = \lim\limits_{x\to1} x+1 = 2$

② $f(1) = (1)+1 = 2$

$\therefore$ f is continuous at $x=1$.

Example 4.1.2

Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

i.e. $x \neq 0$

We have $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} \dfrac{\sin x}{x} = 1$

$\therefore$ f is continuous at $x=0$ $\Leftrightarrow$ $\lim\limits_{x\to0} f(x) = f(0)$, i.e. $a = 1$

Recall :

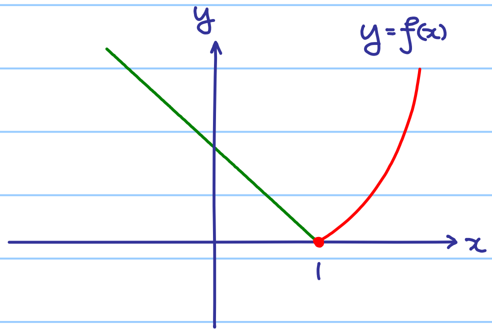
$\lim\limits_{x\to c} f(x) = L$ if and only if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = L$

Rewrite :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$

Example 4.1.3

If $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$



① $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} x^2-1 = 0$

② $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} 1-x = 0$

③ $f(1) = 1^2-1 = 0$

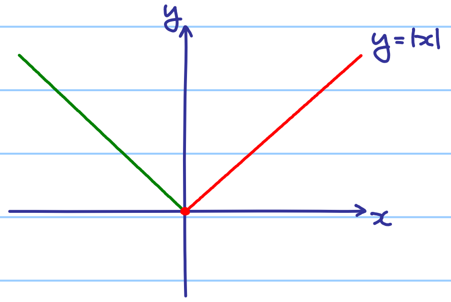
$\therefore$ f is continuous at $x=1$.

Exercise 4.1.1

Show that $f(x)=|x|$ is a continuous function.

Hint: Show that $f(x)$ is continuous

(i) for $x>0$; (ii) at $x=0$; (iii) for $x<0$.



Theorem 4.1.1

- If $f(x)$ and $g(x)$ are continuous at $x=c$, then $f(x)\pm g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$ $(g(c)\neq 0)$ are continuous at $x=c$ as well.
- Polynomial functions and exponential functions are continuous everywhere.
- Trigonometric functions and logarithmic functions are continuous at every point where they are defined.
- If $g(x)$ is continuous at $x=c$ and $f(x)$ is continuous at $x=g(c)$, then $f(g(x))$ is continuous at $x=c$.

(That's why we usually have $\lim_{x\to c} f(x) = f(c)$ as we usually looking at continuous functions.)

Example 4.1.4

Let $f(x) = \frac{2x^2+3}{x^2-3x+2}$ quotient of two polynomials (continuous functions)

$= \frac{2x^2+3}{(x-2)(x-1)}$ the denominator is nonzero when $x\neq 1$ or 2.

$\therefore$ $f(x)$ is continuous at $x\in\mathbb{R}\setminus\{1,2\}$

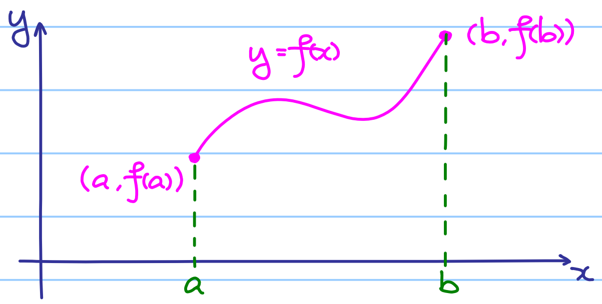
4.2 Continuous on [a,b]

Definition 4.2.1

Let $f : [a,b] \longrightarrow \mathbb{R}$

f is said to be continuous at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$;

f is said to be continuous at $x = b$ if $\lim_{x \to b^-} f(x) = f(b)$.



(We cannot talk about $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$!)

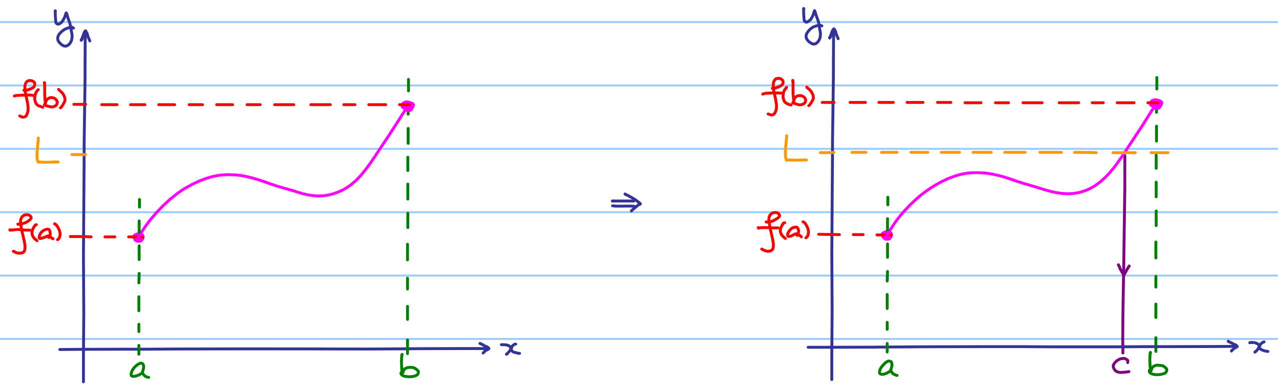
Furthermore, if a function $f : [a,b] \to \mathbb{R}$ is continuous at every point $x \in [a,b]$, then f is said to be continuous on $[a,b]$.

Theorem 4.2.1 (Intermediate Value Theorem)

Suppose that f is continuous on $[a,b]$ and $f(a) < f(b)$.

Furthermore, if $L \in \mathbb{R}$ such that $f(a) < L < f(b)$.

then there exists (at least one) $c \in (a,b)$ such that $f(c) = L$.



Similar result holds for $f(a) > L > f(b)$. (What is the picture ?)

Example 4.2.1

Let $f(x) = x^2$

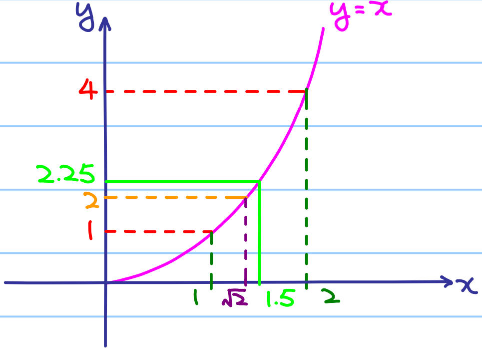
① $f(1) = 1 < 2 < 4 = f(2)$

② f is continuous on $[1,2]$ (In fact, on $\mathbb{R}$)

By the Intermediate Value Theorem, there exists $c \in (1,2)$ such that $f(c) = c^2 = 2$.

c is $\sqrt{2}$ by definition !

. $1 < \sqrt{2} < 2$ (estimates $\sqrt{2}$)



We can further obtain a better estimation by :

① Take the mid-point of $[1,2]$, i.e. 1.5 .

② $f(1.5) = 2.25 > 2$.

③ $f(1) = 1 < 2 < 2.25 = f(1.5)$

$\therefore 1 < \sqrt{2} < 1.5$

Repeating again and again to obtain better and better estimation.

It is well-known as method of bisection !

Example 4.2.2

Show that $2^x = \frac{1}{x^2}$ has at least one solution.

(i.e. let $f(x) = 2^x - \frac{1}{x^2}$, the equation $f(x) = 0$ has at least one solution.)

Note that $f(\frac{1}{2}) = 2^{\frac{1}{2}} - \frac{1}{(\frac{1}{2})^2} = \sqrt{2} - 4 < 0$

$f(1) = 2^1 - \frac{1}{1^2} = 2 - 1 = 1 > 0$

and f is continuous on $[\frac{1}{2}, 1]$.

By the Intermediate Value Theorem, there exists $c \in (\frac{1}{2}, 1)$

such that $f(c) = 2^c - \frac{1}{c^2} = 0$, i.e. $2^c = \frac{1}{c^2}$.

Remark: $\frac{1}{2}$ and 1 can be replaced by other points a and b, but we have

to make sure that f is continuous on $[a,b]$.

$f(-1) = 2^{-1} - \frac{1}{(-1)^2} = \frac{1}{2} - 1 = -\frac{1}{2} < 0$

$f(1) = 2^1 - \frac{1}{1^2} = 2 - 1 = 1 > 0$

Can we use the Intermediate Value Theorem ? No ! f is NOT continuous on $[-1,1]$!

4.3 Relative and Absolute Extrema

Definition 4.3.1

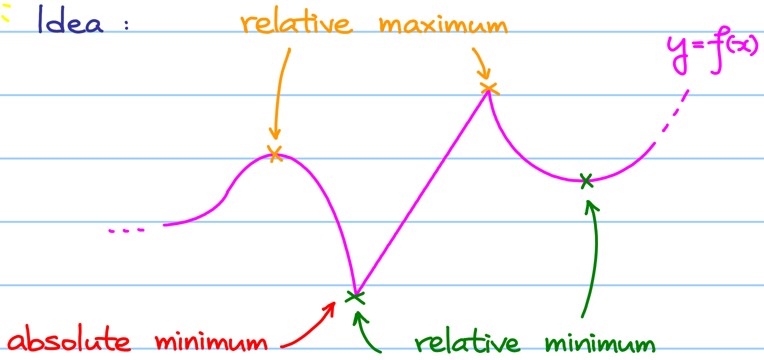
f has an absolute maximum (minimum) point at a if

$f(x) \le f(a)$ $\quad$ ($f(x) \ge f(a)$) for all x in the domain of f.

f has a relative maximum (minimum) point at a if

$f(x) \le f(a)$ $\quad$ ($f(x) \ge f(a)$) for all x in a neighborhood of a.

Idea :



Note : No absolute maximum in this case.

Remark:

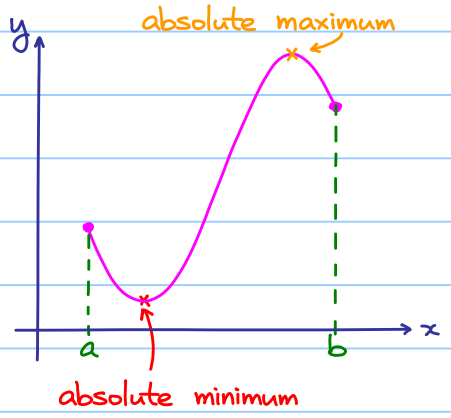
1) We simply use maximum / minimum to refer relative maximum / minimum.

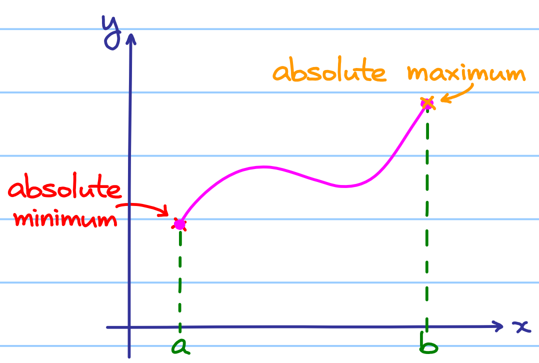
2) Absolute maximum / minimum are also called global maximum / minimum.

Theorem 4.3.1 (Maximum-Minimum Theorem / Extreme-Value Theorem)

Let $f:[a,b] \to \mathbb{R}$ be a continuous function.

Then f has an absolute maximum and an absolute minimum on $[a,b]$.





Absolute maximum / minimum may be attained at the boundary points of $[a,b]$.

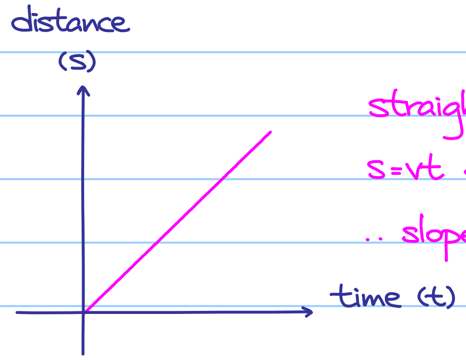
Main question : Given a function, how to find all absolute / relative extrema ?

Differentiation provides a powerful tool for that.

§ 5 Differentiation

5.1 Idea of Derivative

Recall : (average) speed $= \frac{\text{distance}}{\text{time}}$



straight line

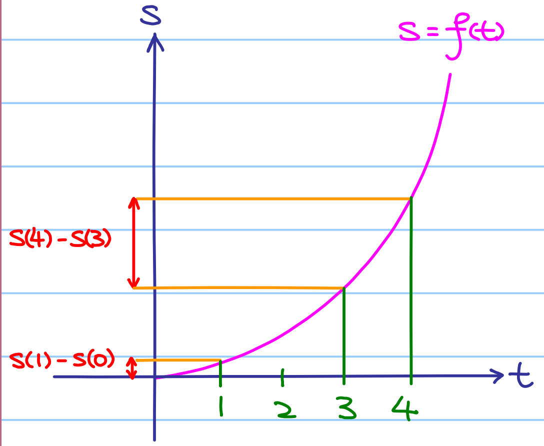
$s = vt \Leftrightarrow v = \frac{s}{t}$

$\therefore$ slope $= v =$ speed .

Remark :

Using displacement and velocity if you know .

How about this case ?



distance traveled from $t=0$ to $t=1$ ($S(1) - S(0)$) < distance traveled from $t=3$ to $t=4$ ($S(4) - S(3)$)

Why? The speed is changing.

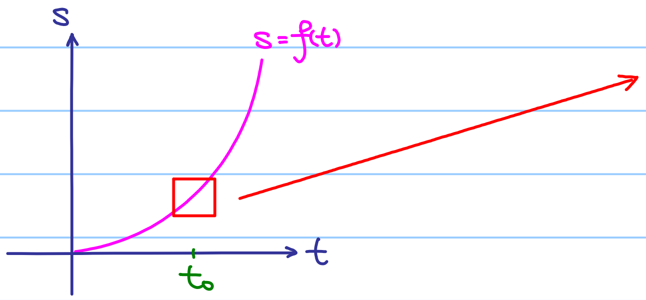
Speed is different at different moment.

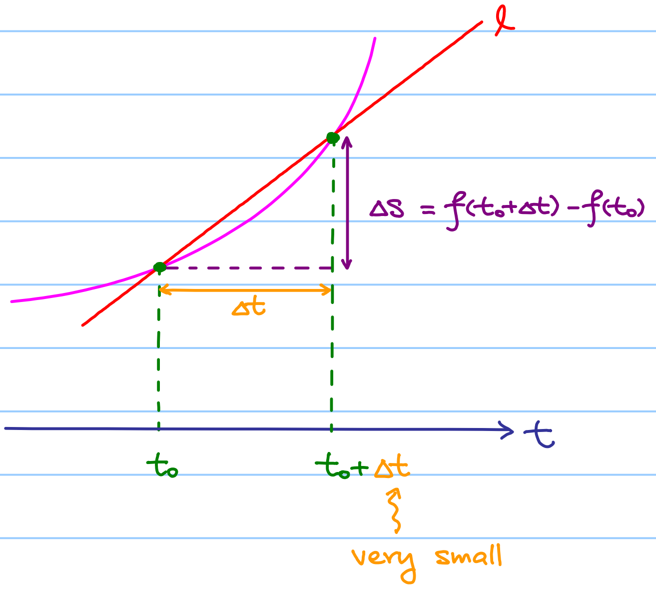
Hold on !

What is the meaning of speed at a particular moment (instantaneous speed) ?

We need a definition !

Instantaneous speed at $t = t_0$:





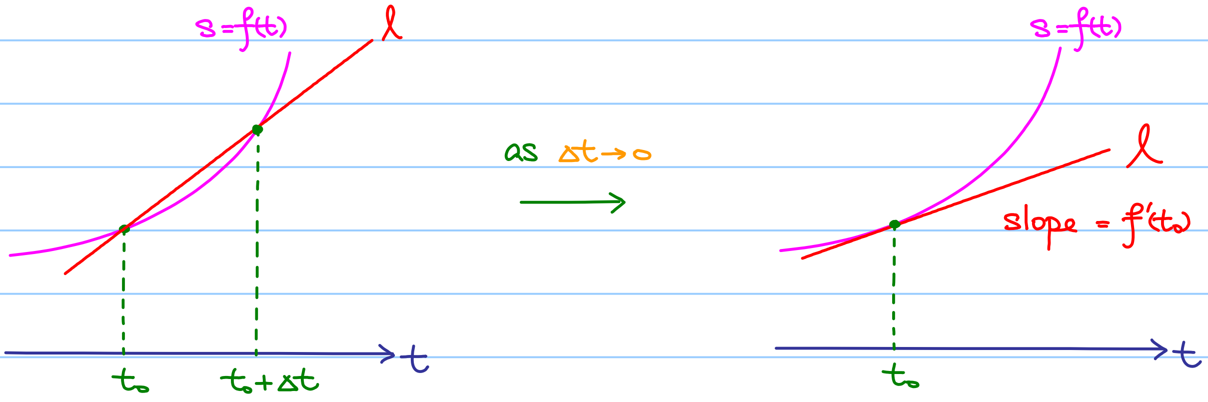
Average speed between t_0 and $t_0 + \Delta t$

$$= \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \text{slope of } \ell$$

Idea : Let Δt becomes smaller and smaller !

Instantaneous speed at $t = t_0$ is defined to be $\lim_{\Delta t \to 0} \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t}$

(provided it exists , if so , it is denoted by $f'(t_0)$)



Note : When $\Delta t \to 0$, ℓ becomes the tangent line at $t = t_0$, so

slope of the tangent line at $t = t_0$ = $f'(t_0)$

Example 5.1.1

If $s = f(t) = t^2$, find $f'(2)$ (instantaneous speed at $t=2$).

$$f'(2) = \lim_{\Delta t \to 0} \frac{f(2+\Delta t) - f(2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(2+\Delta t)^2 - 2^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{2 \cdot 2 \Delta t + \Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 2 \cdot 2 + \Delta t \quad = 2 \cdot 2 = 4$$

In general, we have $y = f(x)$, fix x_0.

Then $f'(x_0)$ means rate of change of y with respect to x at $x = x_0$.

Definition 5.1.1

$f(x)$ is said to be differentiable at $x = x_0$ if $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ exists (called the first principle).

It is called the derivative of $f(x)$ at $x = x_0$ and it is denoted by $f'(x_0)$.

Note: By definition, if $f(x_0)$ is NOT well-defined, then $f'(x_0)$ is NOT well-defined.

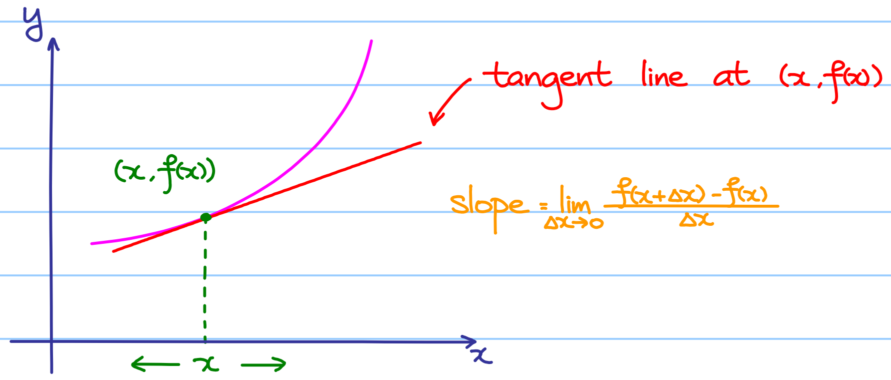
Let $\Delta x = x - x_0$, i.e. $x = x_0 + \Delta x$

When Δx tends to 0, x tends to x_0

Therefore, we have another formulation:

$f(x)$ is said to be differentiable at $x = x_0$ if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.

Perform the previous step at every point:



Recall : What is a function ?

Roughly speaking, given an input x, return a value

Now, we construct a new function, $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ (if exists)

(i.e. given an input x, return the slope of the tangent line at $(x, f(x))$.)

Example 5.1.2

If $f(x) = x^2$, find $f'(x)$.

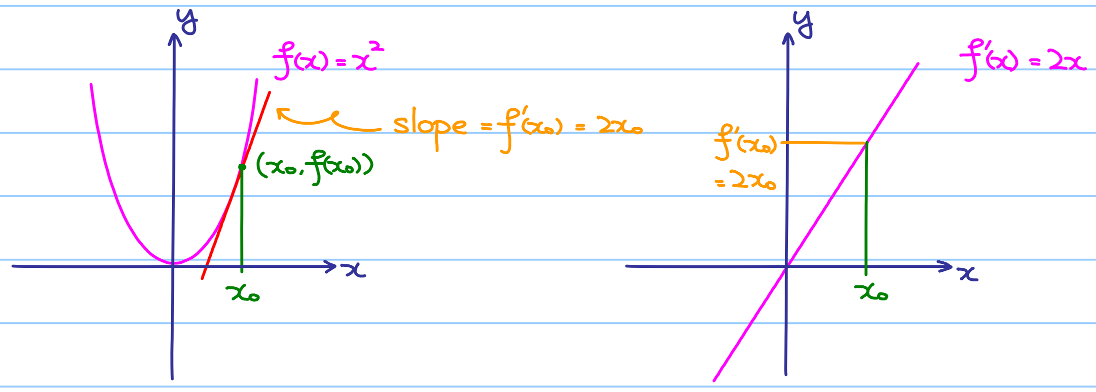
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Relation between the graphs of $f(x) = x^2$ and $f'(x) = 2x$:



Notations :

$$y = f(x) = x^2$$

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x) = 2x$$

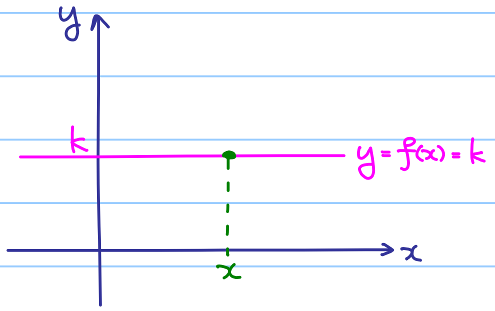
$$\left.\frac{df}{dx}\right|_{x=3} = \left.\frac{dy}{dx}\right|_{x=3} = f'(3) = 2(3) = 6$$

Definition 5.1.2

If $f : A \to \mathbb{R}$ is a function that is differentiable at every point in A, then $f(x)$ is said to be a differentiable function.

Theorem 5.1.1

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.



Note : tangent line at $(x, f(x))$ is horizontal

$\therefore f'(x) = 0$

Concrete computation :

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{k-k}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$
$$= \lim_{\Delta x \to 0} 0$$
$$= 0$$

Exercise 5.1.1

Find $f'(x)$ if

(a) $f(x) = x$ Ans : $f'(x) = 1$

(b) $f(x) = x^3$ $f'(x) = 3x^2$

Theorem 5.1.2

If $f(x) = x^r$, where r is a real number, then $f'(x) = rx^{r-1}$ whenever it is defined.

(Think: If $r = \frac{1}{2}$, $f(x) = \sqrt{x}$ which is defined when $x \geq 0$.)

proof :

We only prove the case $f(x) = x^n$, where n is a natural number.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(C_0^n x^n + C_1^n x^{n-1}\Delta x + C_2^n x^{n-2}\Delta x^2 + \cdots + C_n^n \Delta x^n) - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} C_1^n x^{n-1} + \underbrace{C_2^n x^{n-2}\Delta x + \cdots + C_n^n \Delta x^{n-1}}_{\text{terms with powers of } \Delta x}$$
$$= nx^{n-1}$$

5.2 Differentiability and Continuity

Theorem 5.2.1

If $f(x)$ is differentiable at $x=x_0$, then $f(x)$ is continuous at $x=x_0$

proof : By assumption, $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ exists

Also, we know $\lim\limits_{\Delta x\to 0}\Delta x = 0$

$$\lim_{\Delta x\to 0} f(x_0+\Delta x)-f(x_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\cdot\Delta x$$

both exist

$$= \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\cdot\lim_{\Delta x\to 0}\Delta x$$

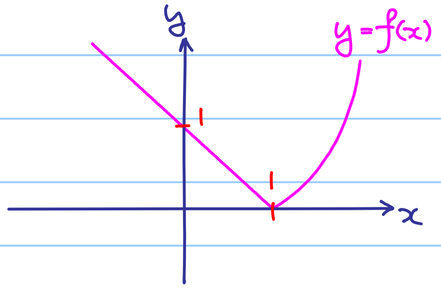
$$= f'(x_0)\cdot 0 = 0$$

$\therefore \lim\limits_{\Delta x\to 0} f(x_0+\Delta x) = f(x_0)$, so $f(x)$ is continuous at $x=x_0$.

However, the converse is NOT true.

Example 5.2.1

$$\text{Let } f(x) = \begin{cases} x^2-1 & \text{if } x\geq 1 \\ 1-x & \text{if } x<1 \end{cases}$$



$$\lim_{\Delta x\to 0^+}\frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x\to 0^+}\frac{[(1+\Delta x)^2-1]-[1^2-1]}{\Delta x} = \lim_{\Delta x\to 0^+}\frac{2\Delta x+\Delta x^2}{\Delta x} = 2$$

(it means we are looking at small but positive Δx)

$$\lim_{\Delta x\to 0^-}\frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x\to 0^-}\frac{[1-(1+\Delta x)]-[1^2-1]}{\Delta x} = \lim_{\Delta x\to 0^-}\frac{-\Delta x}{\Delta x} = -1$$

(it means we are looking at small but negative Δx)

$$\lim_{\Delta x\to 0^+}\frac{f(1+\Delta x)-f(1)}{\Delta x} \neq \lim_{\Delta x\to 0^-}\frac{f(1+\Delta x)-f(1)}{\Delta x} \Rightarrow \lim_{\Delta x\to 0}\frac{f(1+\Delta x)-f(1)}{\Delta x} \text{ does NOT exist !}$$

$\therefore f(x)$ is NOT differentiable at $x=1$.

Exercise 5.2.1

a) Show that $f(x)$ is continuous at $x=1$, i.e. $\lim_{x\to 1} f(x) = f(1)$.

(Therefore, the converse statement of theorem 5.2.1 is NOT true.)

b) Write down $f'(x)$ for $x \neq 1$.

Answer :

$$f'(x) = \begin{cases} 2x & \text{if } x>1 \\ \text{undefined} & \text{if } x=1 \\ -1 & \text{if } x<1 \end{cases}$$

5.3 Elementary Rules of Differentiation

Theorem 5.3.1

If $f(x)$ and $g(x)$ are differentiable functions, then

1) $(f+g)'(x) = f'(x) + g'(x)$

2) $(f-g)'(x) = f'(x) - g'(x)$

3) [product rule] $(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

4) [qustient rule] $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$

proof of (3) :

$$\lim_{\Delta x\to 0} \frac{(f\cdot g)(x+\Delta x) - (f\cdot g)(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \underset{+}{g(x+\Delta x)} + f(x)\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$= f'(x)g(x) + f(x)g'(x)$

+ $g(x)$ is differentiable

$\Rightarrow g(x)$ is continuous

$\Rightarrow \lim_{\Delta x\to 0} g(x+\Delta x) = g(x)$

Direct consequence :

Theorem 5.3.2

If k is a constant and $f(x)$ is a differentiable function, then $(k\cdot f)'(x) = kf'(x)$.

proof :

Using the product rule and theorem 5.1.1

$(k\cdot f)'(x) = \underbrace{(k)'}_{0} f(x) + kf'(x) = kf'(x)$

Example 5.3.1

Find $\frac{d}{dx}(3x^2+7x-2)$

$$\frac{d}{dx}(3x^2+7x-2) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(2)$$
$$= 3\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - \frac{d}{dx}(2)$$
$$= 3(2x) + 7(1) - 0$$
$$= 6x + 7$$

Example 5.3.2

Find $\frac{d}{dx}(3x^2-5x+1)(2x+7)$

$$\frac{d}{dx}[(3x^2-5x+1)(2x+7)]$$
$$= [\frac{d}{dx}(3x^2-5x+1)](2x+7) + (3x^2-5x+1)[\frac{d}{dx}(2x+7)]$$
$$= (6x-5)(2x+7) + (3x^2-5x+1)(2)$$
$$= 18x^2+22x-33$$

Try to compare : Expand $(3x^2-5x+1)(2x+7)$ and get $6x^3+11x^2-33x+7$

Then differentiate , get the same result ?

Example 5.3.3

Find the derivative of $\frac{2x}{x^2+1}$.

$$\frac{d}{dx}\frac{2x}{x^2+1} = \frac{[\frac{d}{dx}(2x)](x^2+1) - (2x)[\frac{d}{dx}(x^2+1)]}{(x^2+1)^2}$$
$$= \frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$$
$$= \frac{-2x^2+2}{(x^2+1)^2}$$

Example 5.3.4

Find the derivative of $\frac{1}{\sqrt{x}} + \sqrt{x}$

$$\frac{d}{dx}(\frac{1}{\sqrt{x}} + \sqrt{x}) = \frac{d}{dx}(x^{-\frac{1}{2}} + x^{\frac{1}{2}})$$

$$= -\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$$

5.4 Higher Derivatives

$s(t)$: distance functions (depends on time t)

(instantaneous) speed = rate of change of distance travelled with respect to t.

$v(t) = \frac{ds}{dt}$ (still a function of t)

Question : What is $\frac{dv}{dt}$?

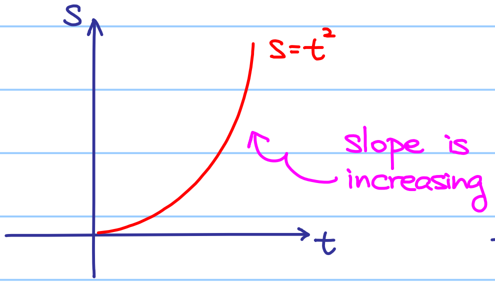
Answer : Acceleration !

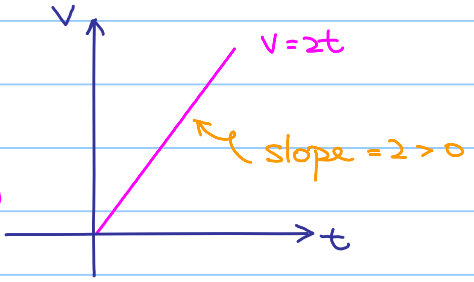
= rate of change of speed with respect to t.

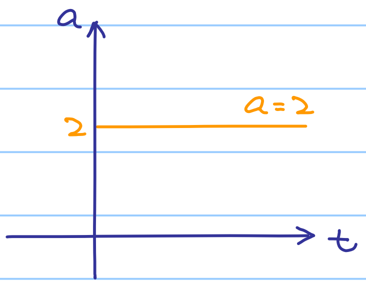
We write $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Example 5.4.1

$s(t) = t^2$ $v(t) = \frac{ds}{dt} = 2t$ $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2$







speed is increasing

i.e. accelerating

Notations :

In general, let $y=f(x)$.

We have : (1st derivative) $\frac{dy}{dx} = \frac{df}{dx} = f'(x)$

(2nd derivative) $\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x)$

(n th derivative) $\frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x)$

5.5 Derivatives of Trigonometric Functions

Preparations :

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2(\frac{x}{2})}{x^2}$$

Note: $\cos x = 1 - 2\sin^2(\frac{x}{2})$

$\therefore \ 1-\cos x = 2\sin^2(\frac{x}{2})$

$$= \lim_{x\to 0} \frac{1}{2} \frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2}$$

$$= \frac{1}{2}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot x$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \lim_{x\to 0} x$$

$$= \frac{1}{2} \cdot 0$$

$$= 0$$

$$\lim_{\Delta x\to 0} \frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim_{\Delta x\to 0} \frac{\cos x \cos\Delta x - \sin x \sin\Delta x - \cos x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \cos x \frac{\cos\Delta x - 1}{\Delta x} - \sin x \frac{\sin\Delta x}{\Delta x}$$

$$= -\sin x$$

$(\because \lim_{\Delta x\to 0} \frac{\cos\Delta x - 1}{\Delta x} = 0$ and $\lim_{\Delta x\to 0} \frac{\sin\Delta x}{\Delta x} = 1 .)$

$\therefore \ \frac{d}{dx} \cos x = -\sin x$

Exercise 5.5.1

Show that $\frac{d}{dx} \sin x = \cos x$ by using method similar to the above.

$\tan x = \frac{\sin x}{\cos x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \csc x = \frac{1}{\sin x}$ $\quad \cot x = \frac{\cos x}{\sin x}$

$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} \overset{(*)}{=} \cdots = \frac{1}{\cos^2 x} = \sec^2 x$ $\quad$ (*) Exercise : By quotient rule

Exercise 5.5.2

Show that

a) $\frac{d}{dx} \sec x = \sec x \tan x$

b) $\frac{d}{dx} \csc x = -\csc x \cot x$

c) $\frac{d}{dx} \cot x = -\csc^2 x$

5.6 Derivative of e^x

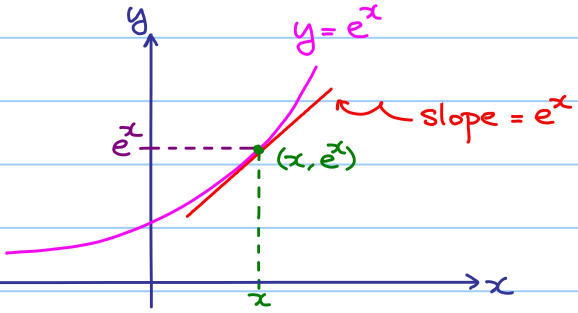
Cheating : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= e^x$$ (getting back itself)

Geometrical meaning :



Example 5.6.1

Find $\frac{d}{dx}[e^x(3x^2+7x-2)]$

$$\frac{d}{dx}[e^x(3x^2+7x-2)] = [\frac{d}{dx}e^x](3x^2+7x-2) + e^x[\frac{d}{dx}(3x^2+7x-2)]$$

$$= e^x(3x^2+7x-2) + e^x(6x+7)$$

$$= e^x(3x^2+13x+5)$$

Question : How do we differentiate a more complicated function , such as $\sqrt{x^2+3x}$?

We need a tool called chain rule

5.7 Chain Rule

Theorem 5.7.1

If $f: B \to \mathbb{R}$ and $g: A \to \mathbb{R}$ are differentiable functions such that $g(A) \subseteq B$, then the composite function $(f \circ g): A \to \mathbb{R}$ defined by $(f \circ g)(x) = f(g(x))$ is differentiable and

$(f \circ g)'(x) = f'(g(x))\, g'(x)$

Hard to understand ? Let's reformulate it as :

Let $u = g(x)$, $y = f(u) = f(g(x))$, then

the chain rule simply means $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Think : $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$

Example 5.7.1

Find the derivative of $\sqrt{x^2+3x}$.

Let $u = g(x) = x^2+3x$, $\qquad \frac{du}{dx} = 2x+3$

$y = f(u) = \sqrt{u}$ $\qquad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$

then $f(g(x)) = \sqrt{x^2+3x}$

By the chain rule , $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \cdot (2x+3)$

$= \underbrace{\frac{1}{2\sqrt{x^2+3x}}}_{f'(g(x))} \cdot \underbrace{(2x+3)}_{g'(x)}$ $\qquad$ put $u = x^2+3x$ back

differentiate f
then put back $g(x)$

Example 5.7.2

Find the derivative of $(3x^2-2x)^{10}$

Let $u = g(x) = 3x^2-2x$ $\qquad \frac{du}{dx} = 6x-2$

$y = f(u) = u$ $\qquad \frac{dy}{du} = 10u^9$

then $y = f(g(x)) = (3x^2-2x)^{10}$

By chain rule , $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 10u^9 \cdot (6x-2)$

$= 10(3x^2-2x)^9 \cdot (6x-2)$ $\qquad$ put $u = 3x^2-2x$ back

$= 20(3x^2-2x)^9 \cdot (3x-1)$

Slogan : differentiate layer by layer .

Exercise 5.7.1

Show that $\frac{d}{dx} e^{ax} = ae^{ax}$

Show that $\frac{d}{dx} a^x = (\ln a) a^x$, for $a > 0$. (Hint : $a^x = e^{(\ln a^x)} = e^{(\ln a)x}$)

Exercise 5.7.2

Find the derivative of $\left(\frac{x}{x+1}\right)^2$

a) by using the chain rule ;

b) by writing $\left(\frac{x}{x+1}\right)^2 = \frac{x^2}{(x+1)^2}$ and using the quotient rule.

Answer : Both equal to $\frac{2x}{(x+1)^3}$.

Example 5.7.3

Find the derivative of $e^{\sqrt{x^2+1}}$.

1st layer $y = e^w$ $\quad w = \sqrt{x^2+1}$

2nd layer $w = \sqrt{u}$ $\quad u = x^2+1$

3rd layer $u = x^2+1$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

Example 5.7.3

Revisit of quotient rule :

$$\left(\frac{f}{g}\right)'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\,[g(x)]^{-1}\right)$$

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\frac{d}{dx}[g(x)]^{-1}$$

Apply the chain rule

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\left\{-[g(x)]^{-2}\frac{dg}{dx}\right\}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

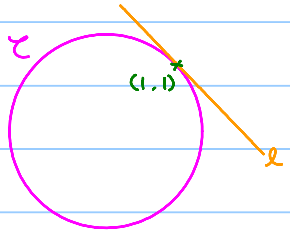
5.8 Implicit Differentiation

Example 5.8.1

$x^2+y^2=2$ —— $\mathcal{C}$

Locus of $\mathcal{C}$ is a circle centered at $(0,0)$ with radius $\sqrt{2}$.

Check: $(1,1)$ is a point lying on the circle.



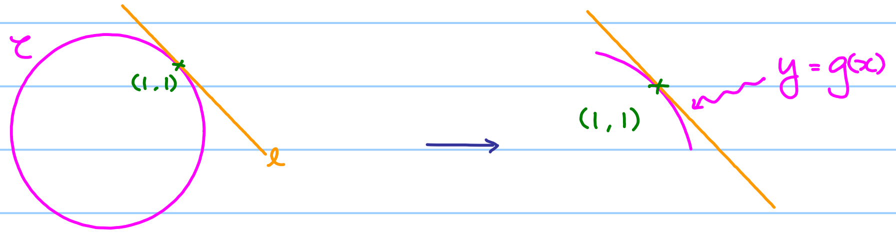
We want to find the equation of the tangent line ℓ

(i.e. need to know the slope of ℓ)

Note: $x^2+y^2=2$ is NOT a function!

Question: How to find $\frac{dy}{dx}$? (Actually, is it well defined ?)

Answer: Yes, roughly speaking.



The small segment of $\mathcal{C}$ containing $(1,1)$ can be regarded as the graph of some function $y=g(x)$. (In fact, $g(x)=\sqrt{2-x^2}$ in this case.)

How to find ? Do it as usual !

$$x^2+y^2=2$$

differentiate both sides with respect to x.

$$2x+\frac{d}{dx}y^2=0$$

$$2x+2y\frac{dy}{dx}=0 \qquad \text{(Applying chain rule)}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

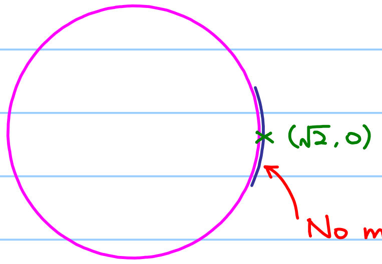
$\therefore \frac{dy}{dx}=-1$ when $(x,y)=(1,1)$.

We denote it by $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=-1$

Remark.

$\frac{dy}{dx}$ is defined at a point of a curve only if a small arc containing the point can be regarded as the graph of some function $y=g(x)$.

$\therefore$ $\frac{dy}{dx}$ is NOT defined when $(x,y)=(\pm\sqrt{2},0)$.



No matter how small the arc is,

it cannot be realized as graph of some function $y=g(x)$.

Implicit differentiation : Apply differentiation to $F(x,y)=0$.

$$\text{e.g. } x^2+y^2=2 \longrightarrow \underbrace{x^2+y^2-2}_{F(x,y)}=0$$

Example 5.8.2

Let $\mathcal{C}$ be the curve defined by the equation $x^3+2y^3+2xy=5$.

Show that $P(1,1)$ is a point lying on $\mathcal{C}$.

Find the equation of the tangent line of $\mathcal{C}$ at P.

Put $(x,y)=(1,1)$, $LHS=(1)^3+2(1)^3+2(1)(1)=5=RHS$

$\therefore$ $P(1,1)$ lies on $\mathcal{C}$.

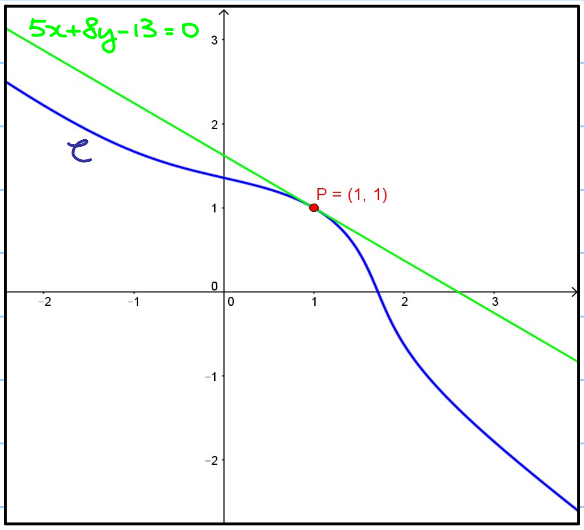
$$x^3+2y^3+2xy=5$$

$$3x^2+6y^2\frac{dy}{dx}+2\left(y+x\frac{dy}{dx}\right)=0$$

$$(3x^2+2y)+(6y^2+2x)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{3x^2+2y}{6y^2+2x}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=-\frac{5}{8}$$



the equation of the tangent line of $\mathcal{C}$ at P :

$$\frac{y-1}{x-1}=-\frac{5}{8}$$

$$5x+8y-13=0$$

Applications :

Example 5.8.3

Differentiation of Logarithmic Function

Let $y = \ln x$, $x > 0$. Then $e^y = x$,

differentiate both sides with respect to x .

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$.

Exercise 5.8.1

By rewriting $\log_a x = \frac{\ln x}{\ln a}$, show that $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$.

Example 5.8 4

Let $y = \ln|x|$, $x \neq 0$. Find $\frac{dy}{dx}$.

We can rewrite $y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

For $x > 0$, we have just shown that $\frac{dy}{dx} = \frac{1}{x}$

For $x < 0$,

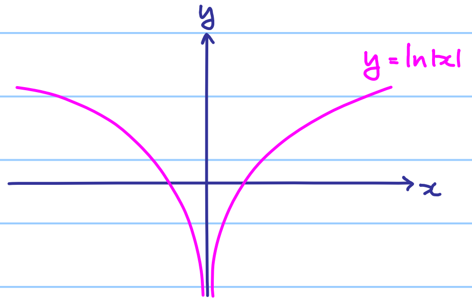
$$y = \ln(-x)$$

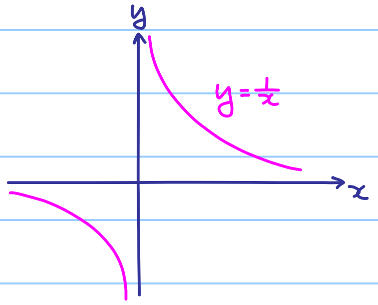
$$e^y = -x$$

$$e^y \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln|x| = \frac{1}{x}$ for $x \neq 0$.





Note: It is why $\int \frac{1}{x} dx = \ln|x| + C$.

Example 5.8.5

If $y = \sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}$, then find $\frac{dy}{dx}$.

Difficult to differentiate by using chain rule and quotient rule!

$$y = \frac{(x-1)^{\frac{1}{3}}(x-2)^{\frac{2}{3}}}{(x-4)^{\frac{2}{3}}}$$

$$|y| = \frac{|x-1|^{\frac{1}{3}}|x-2|^{\frac{2}{3}}}{|x-4|^{\frac{1}{3}}}$$

$$\ln|y| = \ln\frac{|x-1|^{\frac{1}{3}}|x-2|^{\frac{2}{3}}}{|x-4|^{\frac{1}{3}}}$$

$$= \frac{1}{3}\ln|x-1| + \frac{2}{3}\ln|x-2| - \frac{1}{3}\ln|x-4|$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3(x-1)} + \frac{2}{3(x-2)} - \frac{1}{3(x-4)}$$ (Apply implicit differentiation)

$$\frac{dy}{dx} = \frac{y}{3}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right)$$

$$= \frac{1}{3}\sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right)$$

Example 5.8 6

Let $y = \frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$ Find $\frac{dy}{dx}$.

$$y = \frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$$

$$\ln y = 5x + \frac{1}{3}\ln(x^2+1) - 4\ln(3x^2+1)$$

Ex : ⋮

Ans : $\frac{dy}{dx} = \left[5 + \frac{2x}{3(x^2+1)} - \frac{24x}{3x^2+1}\right]\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$

Example 5.8 7

Differentiation of Inverse Trigonometric Functions

Let $y = \sin^{-1}x$, $\sin^{-1} : [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, $\sin y = x$.

differentiate both sides with respect to x.

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\sin y = x$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$= \sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

$$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

(rejected , $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \Rightarrow \cos y \geq 0$)

Let $y = \cos^{-1} x$, $\cos^{-1} : [-1,1] \longrightarrow [0,\pi]$. Then , $\cos y = x$.

differentiate both sides with respect to x .

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sin y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$\cos y = x$, $0 \le y \le \pi$

$$\sin y = \pm\sqrt{1-\cos^2 y}$$

$$= \sqrt{1-x^2} \quad \text{or} \quad -\sqrt{1-x^2}$$

$\therefore \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$ (rejected , $0 \le y \le \pi \Rightarrow \sin y \ge 0$)

Exercise 5.8.1

Let $y = \tan^{-1} x$, $\tan^{-1} : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.

Find $\frac{dy}{dx}$ Ans : $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

Example 5.8.8

Let $y = x^x$, $x > 0$. Find $\frac{dy}{dx}$.

Note : The power is NOT a constant, we cannot use the formular $\frac{d}{dx} x^n = nx^{n-1}$.

$$y = x^x$$

$$\ln y = \ln x^x = x \ln x$$

differentiate both sides with respect to x .

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\frac{dy}{dx} = (\ln x + 1) y$$

$$= (\ln x + 1) x^x$$

5.9 Revisit of Parametric Curves in $\mathbb{R}^2$

Let I be an interval and let $\gamma \cdot I \to \mathbb{R}^2$ be a parametric curve such that $\gamma(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable functions.

If γ represents the locus of a moving particle and t represents time , then

(1) $\gamma'(t) := (x'(t), y'(t))$ is the velocity and $|\gamma'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is the speed of the moving particle,

(2) By chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, assume $x'(t) = \frac{dx}{dt} \neq 0$, then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$.

Example 5.9.1

Let $\gamma(t)=(x(t),y(t))=(t^3,t^2)$, for $t>0$, be the locus of a moving particle.

a) Find the velocity and speed of the moving particle.

b) Find $\frac{dy}{dx}$ in terms of t by using $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'(t)}{x'(t)}$.

c) Write down an equation in x and y to describe the locus of the moving particle.

Hence, find $\frac{dy}{dx}$ and compare with the result in (b).

a) velocity of the moving particle $=\gamma'(t)=(x'(t),y'(t))=(3t^2,2t)$

speed of the moving particle $=|\gamma'(t)|=\sqrt{[x'(t)]^2+[y'(t)]^2}=\sqrt{4t^2+9t^4}$

b) $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'(t)}{x'(t)}=\frac{2t}{3t^2}=\frac{2}{3t}$ (Note: Since $t>0$, $x'(t)=3t^2>0$)

c) We have
$$\begin{cases} x=t^3 & \text{—— (1)}. \\ y=t^2 & \text{—— (2)} \end{cases}$$

Eliminating t by considering $(1)^2$ and $(2)^3$, we get $y^3=x^2$,

so $y=\sqrt[3]{x^2}=x^{\frac{2}{3}}$

We have $\frac{dy}{dx}=\frac{2}{3}x^{-\frac{1}{3}}$

Remark: $x=t^3$, so $x^{-\frac{1}{3}}=t$ and we can see results in both (b) and (c) agree.

Exercise 5.9.1

Let $a,b>0$ and let $\gamma(t)=(x(t),y(t))=(a\cos t,b\sin t)$ for $t\in[0,2\pi]$,

be the locus of a moving particle.

a) Find the velocity and speed of the moving particle.

b) Find $\frac{dy}{dx}$ in terms of t by using $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'(t)}{x'(t)}$.

c) Write down an equation in x and y to describe the locus of the moving particle.

Ans: a) velocity $=(-a\sin t, b\cos t)$ and speed $=\sqrt{a^2\sin^2 t+b^2\cos^2 t}$

b) $\frac{dy}{dx}=-\frac{b}{a}\cot t$

c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

§6 Applications of Differentiation

6.1 Rolle's Theorem and Mean Value Theorem

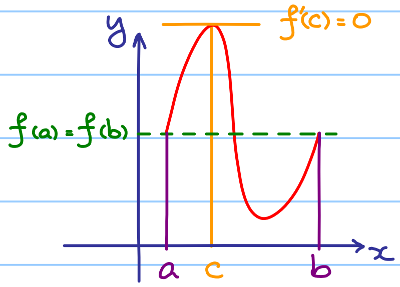
Theorem 6.1.1 (Rolle's Theorem)

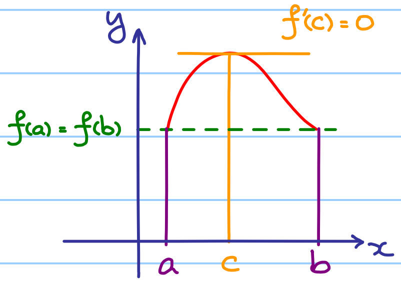
Let $f : [a,b] \rightarrow \mathbb{R}$ be a function such that

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)
3) $f(a) = f(b)$

then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Geometrical meaning:





Theorem 6.1.2 (Mean Value Theorem)

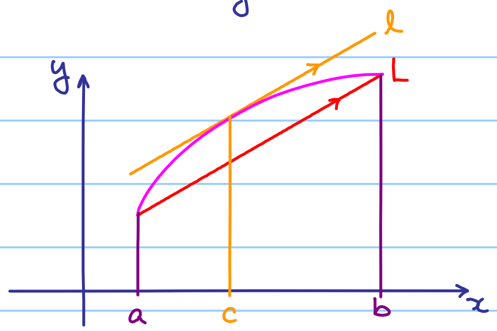
Let $f : [a,b] \rightarrow \mathbb{R}$ be a function such that

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)

then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

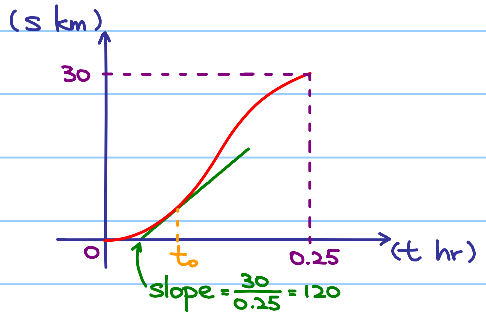
slope of l slope of L.

Geometrical meaning:



Question :

A vehicle is speeding on a highway if its speed ≥ 120 km/hr (at some moment)
If the length of the highway is 30 km and if a driver only spent 15 minutes on the highway. Should he be arrested ?



By the MVT, there exists $t_0 \in (0, 0.25)$
such that slope of the tangent at $t = t_0$ $= \frac{30}{0.25} = 120$
i.e. instantaneous speed at $t = t_0$ = 120 km/hr

6.2 Applications of Mean Value Theorem

Theorem 6.2.1

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable and $f'(x) = 0 \quad \forall x \in \mathbb{R}$,
then $f(x)$ is a constant function

proof : Fix $x_0 \in \mathbb{R}$, let $x \in \mathbb{R} \setminus \{x_0\}$

If $x > x_0$, note f is differentiable everywhere (in particular, on (x_0, x))
$\Rightarrow f$ is continuous everywhere (in particular, on $[x_0, x]$)

Apply MVT, $\exists\, c \in (x_0, x)$ such that

$$f(x_0) - f(x) = \underbrace{f'(c)}_{\overset{\|}{0}\text{ by assumption.}} (x - x_0) = 0$$

i.e. $f(x) = f(x_0) \quad \forall x > x_0$

We have similar result if $x < x_0$, the result follows.

Example 6.2.1

Let $f(x) = \cos^2 x + \sin^2 x$

$f'(x) = -2\cos x \sin x + 2 \sin x \cos x = 0$

$\therefore$ $\cos^2 x + \sin^2 x$ is a constant.

In particular, $f(0) = 1$, so $f(x) = \cos^2 x + \sin^2 x = 1$

Theorem 6.2.2

If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in \mathbb{R}$, then $f(x) = g(x) + C$, where C is a constant.

proof: Let $h(x) = f(x) - g(x)$

Then $h'(x) = f'(x) - g'(x) = 0$

$\therefore h(x) = C$, where C is a constant. i.e. $f(x) = g(x) + C$.

Next, we are going to discuss how differentiation helps to find maximum / minimum points of a function.

Firstly, we make some preparations:

6.3 Increasing / Decreasing Functions

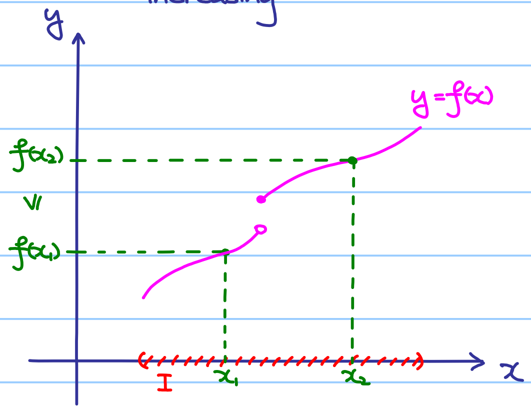
Definition 6.3.1

Let I be an interval and let $f : I \to \mathbb{R}$ be a function such that

$f(x_1) \le f(x_2)$ $(f(x_1) \ge f(x_2))$ for all $x_1 < x_2$.

then $f(x)$ is called an increasing (a decreasing) function.†

Increasing



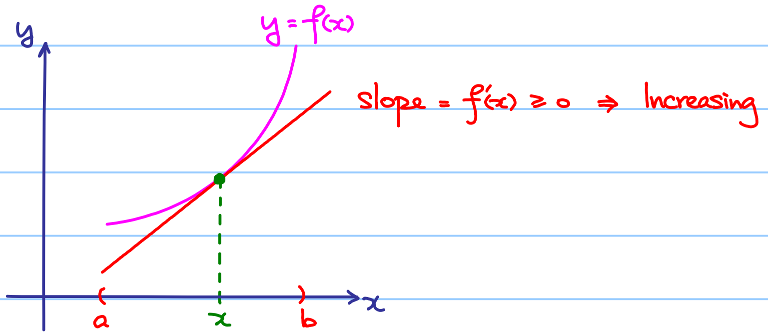
Roughly speaking:
The larger x we input
the larger y we get!

† If we have a strictly inequality, it is called a strictly increasing (decreasing) function.

Theorem 6.3.1

Let $f:(a,b)\to\mathbb{R}$ be a differentiable function.

If $f'(x)\geq 0$ ($f'(x)\leq 0$) for all $x\in(a,b)$ then f is increasing (decreasing) on (a,b). ††



†† If we have strict inequality, then $f(x)$ is strictly increasing (decreasing) on (a,b).

proof :

If $a<x_1<x_2<b$,

apply the MVT to f on $[x_1,x_2]$,

$$\exists\, c\in(x_1,x_2) \text{ such that } f(x_2)-f(x_1) = \underbrace{f'(c)}_{\substack{\rotatebox{-90}{$\geq$}\\ 0}}\,\underbrace{(x_2-x_1)}_{\substack{\vee\\ 0}} \geq 0$$

By assumption

A small modification leads the following :

Theorem 6.3.2

Let $f:[a,b]\to\mathbb{R}$ be a function such that

1) f is continuous on $[a,b]$

2) f is differentiable on (a,b) and $f'(x)\geq 0$ ($f'(x)\leq 0$) for all $x\in(a,b)$

Then f is increasing (decreasing) on $[a,b]$.

proof :

If $a\leq x_1<x_2\leq b$,

apply the MVT to f on $[x_1,x_2]$,

$$\exists\, c\in(x_1,x_2) \text{ such that } f(x_2)-f(x_1) = \underbrace{f'(c)}_{\substack{\rotatebox{-90}{$\geq$}\\ 0}}\,\underbrace{(x_2-x_1)}_{\substack{\vee\\ 0}} \geq 0$$

By assumption

6.4 First Derivative Check

Theorem 6.4.1 (First Derivative Check)

Let I be an open interval and let $a \in I$.

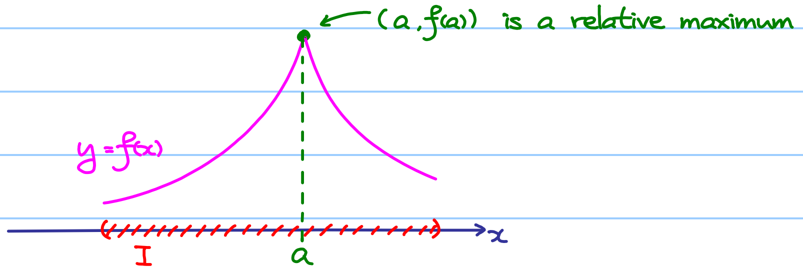
Let $f : I \to \mathbb{R}$ be a function such that

1) f is continuous
2) $f'(x) \geq 0$ $(f'(x) \leq 0)$ for all $x \in I$ with $x < a$
3) $f'(x) \leq 0$ $(f'(x) \geq 0)$ for all $x \in I$ with $x > a$

Then $(a, f(a))$ is a relative maximum (minimum).

Note : We do NOT require the differentiability of f at $x = a$, but only the continuity of f at $x = a$.

Geometrical meaning :



Remember the slogan : Change of sign of $f'(x)$ at $x = a$

Definition 6.4.1

If $f'(a) = 0$, then $(a, f(a))$ is said to be a critical point or stationary point.

Theorem 6.4.2

Let $f : (a,b) \to \mathbb{R}$ be a function and $c \in (a,b)$ such that

1) $f'(c)$ exists
2) f attains maximum (or minimum) at $x = c$

Then, we have $f'(c) = 0$.

Explanation : If $f(x)$ is differentiable everywhere,
then all maximum and minimum points are stationary points.

However, a stationary point is NOT necessary to be a maximum and minimum point!

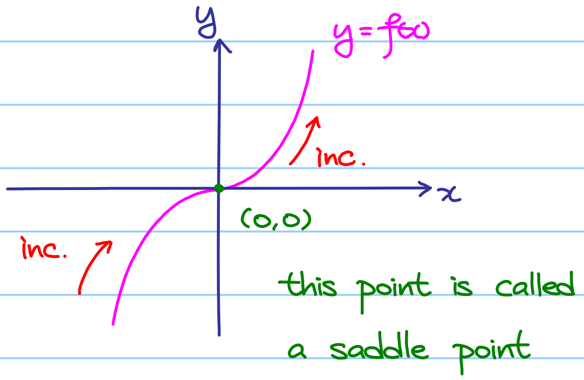
Example 6.4.1

If $f(x) = x^3$, then $f'(x) = 3x^2$

Note : 1) $f'(0) = 0$

2) $f'(x) = 3x^2 > 0$ for $x \neq 0$

i.e. No change of sign of $f'(x)$ at $x = 0$.



this point is called a saddle point

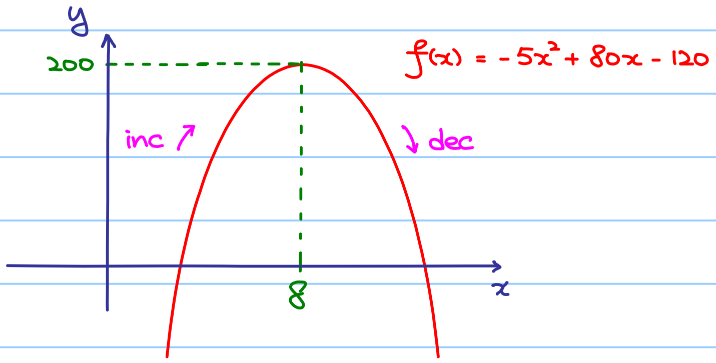
Note : a stationary point is NOT necessary to be a max / min. point !

Example 6.4.2

$f(x) = -5x^2 + 80x - 120$

Note f(x) is a polynomial which is differentiable everywhere.

$$f'(x) = 0$$
$$-10x + 80 = 0$$
$$x = 8$$



$f'(x) > 0$	$f'(x) < 0$
$-10x + 80 > 0$	$-10x + 80 < 0$
$x < 8$	$x > 8$

$\therefore$ f(x) is strictly increasing when $x < 8$ and

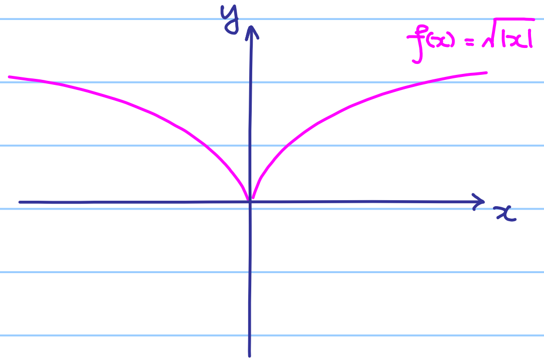
f(x) is strictly decreasing when $x > 8$.

$\therefore$ f(x) attains maximum when $x = 8$ and maximum value $= f(8) = 200$

Remark : Verify the answer by using completing square

Example 6.4.3

Let $f(x) = \sqrt{|x|}$



Rewrite :

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$

If $x > 0$, $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}} > 0$

If $x < 0$, $f(x) = \sqrt{-x}$, then $f'(x) = -\frac{1}{2\sqrt{-x}} < 0$

$\therefore$ $f(x)$ is strictly increasing when $x > 0$

$f(x)$ is strictly decreasing when $x < 0$

However, $\lim_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\sqrt{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{1}{\sqrt{\Delta x}}$ which does NOT exist ,

$\Rightarrow$ $\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$ does NOT exist

$\Rightarrow$ $f'(0)$ does NOT exist

However, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0$

$f(0) = 0$

$\therefore$ $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0) = 0$ and so f is continuous at $x = 0$

By the first derivative check , $f(x)$ attains minimum at $x = 0$.

Example 6.4.4

Prove that $e^x \geq 1+x$ (i.e. $e^x - x - 1 \geq 0$) for all $x \in \mathbb{R}$.

Let $f(x) = e^x - x - 1$

(Want to find the global minimum of $f(x)$ and see if it is ≥ 0.)

$f'(x) = e^x - 1$

$f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$

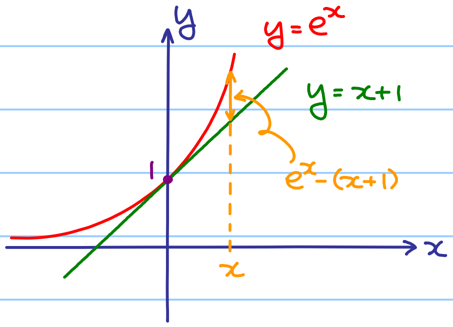
f is strictly increasing when $x > 0$ and strictly decreasing when $x < 0$

(and f is continuous at $x = 0$.)

f attains minimum when $x = 0$ (By 1st derivative check)

(In fact, global minimum, why?)

$$\therefore f(x) \geq f(0) \quad \forall x \in \mathbb{R}$$
$$= e^0 - 0 - 1$$
$$= 0$$



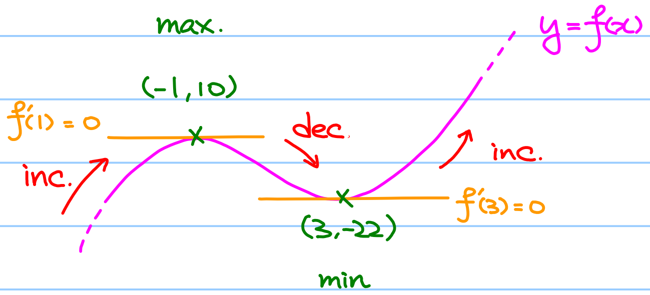
Example 6.4.5

If $f(x) = x^3 - 3x^2 - 9x + 5$

then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$

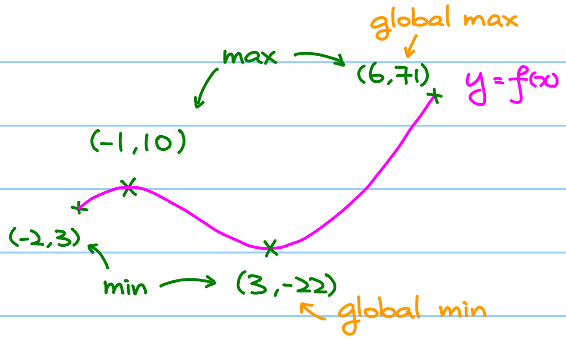


No global max / min in this case

Example 6.4.5

If $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \leq x \leq 6$

$f(-2) = 3$, $f(6) = 59$



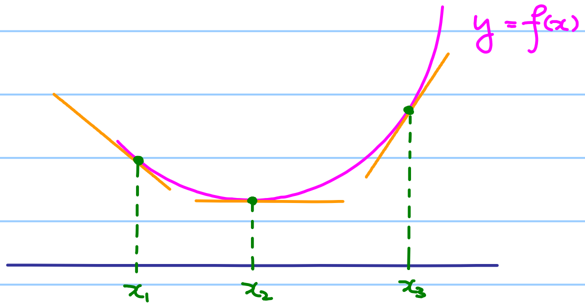
Check the endpoints!

6.5 Second Derivative Check

Let I be an open interval.

$f''(x) > 0$ for $x \in I \Rightarrow f'(x)$ is strictly increasing.

Geometrical meaning:



Slope of the tangent line at $(x, f(x))$ increases as x increases!

(NOT $f(x)$ is increasing!)

Theorem 6.5.1

Let I be an open interval.

If $f''(x) > 0$ ($f''(x) < 0$) for all $x \in I$, then $f(x)$ is concave up (down) on I.

Theorem 6.5.2 (Second Derivative Check)

Let I be an open interval and let $a \in I$

If $f: I \to \mathbb{R}$ be a function such that

1) $f'(a) = 0$ (i.e. $(a, f(a))$ is a stationary point.)

2) $f''(a) < 0$ ($f''(a) > 0$) and $f''(x)$ is continuous at $x = a$ (i.e. f is concave down (up) near $x = a$)

then $(a, f(a))$ is a relative maximum (minimum).

Remark: Actually, the assumption $f''(x)$ is continuous at $x = a$ can be dropped

Caution: If $f''(a) = 0$, then NO conclusion!

Consider $f(x) = x^4, x^3, -x^4$

We have $f'(0) = f''(0) = 0$ in each case, but $(0,0)$ is

- min. for the 1st case
- saddle point for the 2nd case.
- max. for the 3rd case

Example 6.5 1

If $f(x) = x^3 - 3x^2 - 9x + 5$

then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

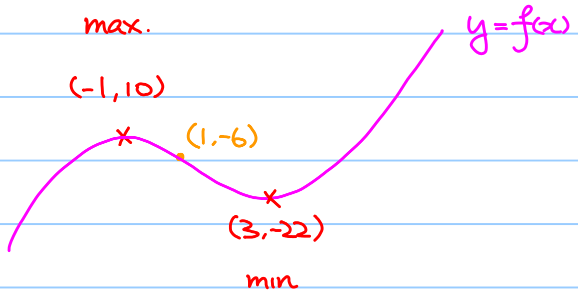
$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$

$f''(x) = 6x - 6$

$f''(x) > 0$ if $x > 1$ $\qquad f''(-1) = 12 < 0$

$f''(x) < 0$ if $x < 1$ $\qquad f''(3) = 12 > 0$



		-1		3	
$f'(x)$	+ve		-ve		+ve
$f(x)$	inc.		dec		inc.

		1	
$f''(x)$	-ve		+ve
$f(x)$	concave down		concave up

Note: The curve changes from being concave down to concave up at (1,6).

This point is called a point of inflection.

Theorem 6.5.1

Let I be an open interval and let $a \in I$.

Let $f: I \to \mathbb{R}$ be a function such that

1) f is continuous

2) $f''(x) > 0$ $(f''(x) < 0)$ for all $x \in I$ with $x < a$

3) $f''(x) < 0$ $(f''(x) > 0)$ for all $x \in I$ with $x > a$

then $(a, f(a))$ is said to be a point of inflection.

Remember the slogan: Change of sign of $f''(x)$ at $x = a$

Example 6.5.2

$f(x) = 12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 120$

Find the range of x such that

(1) $f'(x) > 0$, $f'(x) < 0$

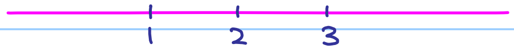
(2) $f''(x) > 0$, $f''(x) < 0$

Step 1: Find $f'(x)$ and factorize it.

$$f'(x) = 60x^4 - 420x^3 + 1020x^2 - 1020x + 360$$

$$= 60(x^4 - 7x^3 + 17x^2 - 17x + 6)$$

$$= 60(x-1)^2(x-2)(x-3)$$ (Using factor theorem)

Step 2: 

gives intervals $x<1$, $1<x<2$, $2<x<3$, $x>3$

Reason: those factors may change sign at the boundary points of intervals.

Step 3:

	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$(x-1)^2$	+	0	+	+	+	+	+
$(x-2)$	−	−	−	0	+	+	+
$(x-3)$	−	−	−	−	−	0	+
$f'(x)$	+	0	+	0	−	0	+
$f(x)$	inc	saddle pt.	inc.	max.	dec.	min	inc.

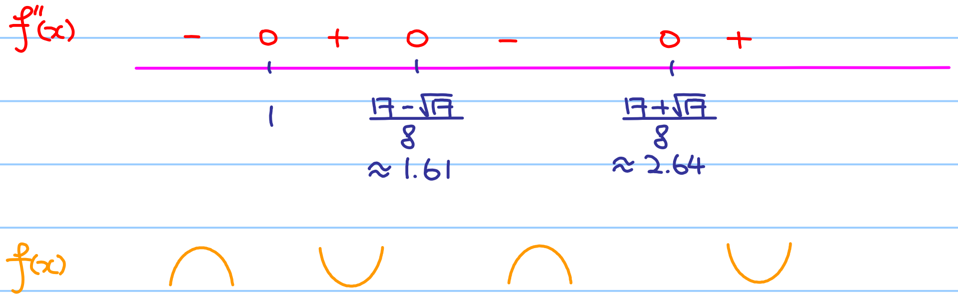
saddle point $= (1, -23)$ max $= (2, -16)$ min $= (3, -39)$

Similarly,

$$f''(x) = 240x^3 - 1260x^2 + 2040x - 1020$$

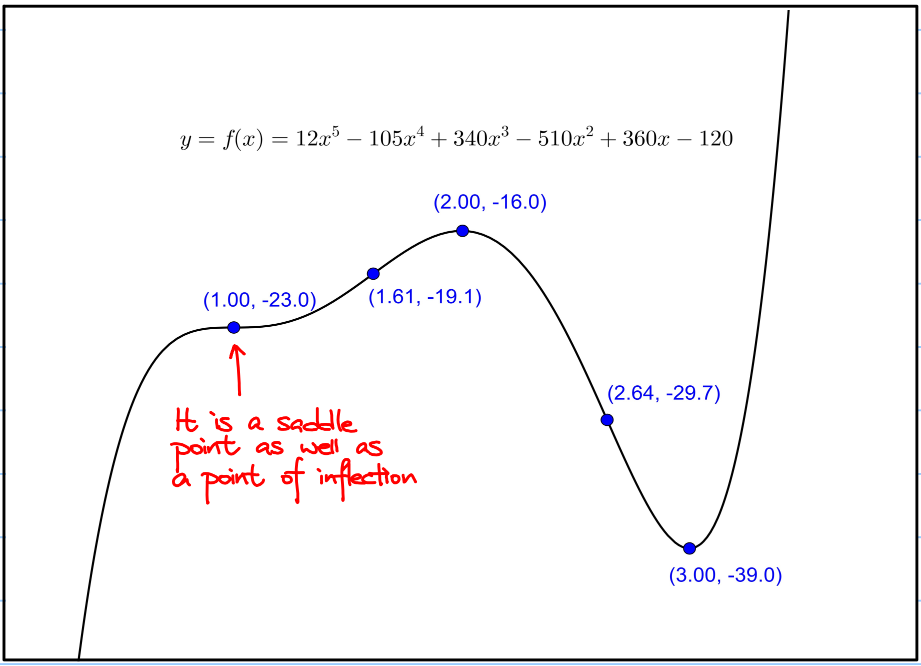
$$= 60(x-1)(4x^2-17x+17)$$

$$= 240(x-1)\left[x-\left(\frac{17+\sqrt{17}}{8}\right)\right]\left[x-\left(\frac{17-\sqrt{17}}{8}\right)\right]$$



points of inflection : $(1,-23)$, $\left(\frac{17\pm\sqrt{17}}{8}, f\left(\frac{17\pm\sqrt{17}}{8}\right)\right)$

$= (1.61, -19.1)$ or $(2.64, -29.7)$



It is a saddle point as well as a point of inflection

Example 6.5.3

$f(x) = \frac{x}{(x+1)^2}$, $x \neq -1$.

$f'(x) = \frac{1-x}{(x+1)^3}$

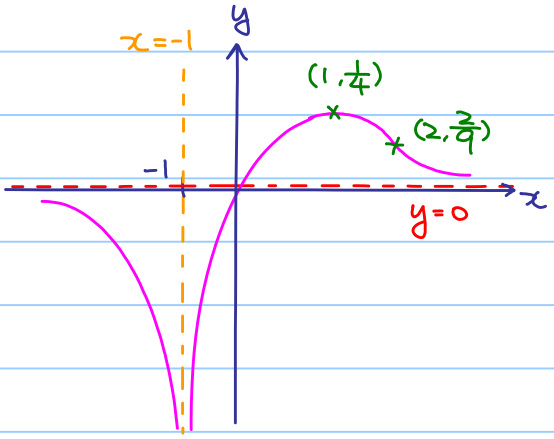
		-1		1	
$f'(x)$	$-$	NOT defined	$+$	0	$-$
⇓					
$f(x)$	dec.	NOT defined	inc.	max.	dec.

max $= (1, \frac{1}{4})$

$f''(x) = \frac{2(x-2)}{(x+1)^4}$

		-1		2	
$f''(x)$	$-$	NOT defined	$-$		$+$
⇓					
$f(x)$	∩		∩		∪

point of inflection : $(2, \frac{2}{9})$



Note : The graph of $y=f(x)$ behaves like :

- the vertical line $x=-1$, when x is "near" -1 .
- the horizontal line $y=0$, when x is "near $+\infty$ or $-\infty$".

In fact, $x=-1$ is called a vertical asymptote,

$y=0$ is called a horizontal asymptote .

6.6 Asymptotes

Definition 6.6.1

1) If $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x) = +\infty$ or $-\infty$,

then $x=a$ is said to be a vertical asymptote.

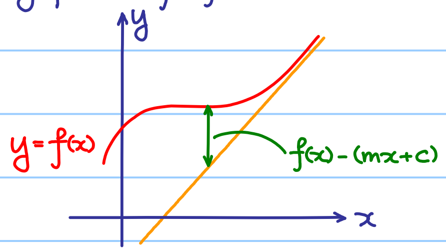
2) If $\lim_{x\to +\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$, where $L \in \mathbb{R}$,

then $y=L$ is said to be a horizontal asymptote.

Note: It may happen that both $\lim_{x\to +\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ exist but they are NOT the same.

3) If $y=mx+c$ is a straight such that $\lim_{x\to +\infty} f(x)-(mx+c)=0$ or $\lim_{x\to -\infty} f(x)-(mx+c)=0$,

then the straight line is said to be an oblique asymptote of $f(x)$.



the distance tends to 0 as $x \to +\infty$

Example 6.6.1

Let $f(x) = \frac{x^2+3x-7}{x-3}$, $x \neq 3$

$f'(x) = \frac{x^2-6x-2}{(x-3)^2}$

		$3-\sqrt{11}$		3		$3+\sqrt{11}$	
$f'(x)$	+	0	−	NOT defined		0	+
⇓				−			
$f(x)$	inc.	max.	dec.	NOT defined	dec.	min.	inc.

max $= (3-\sqrt{11}, f(3-\sqrt{11})) = (3-\sqrt{11}, 9-2\sqrt{11})$ min $= (3+\sqrt{11}, f(3+\sqrt{11})) = (3+\sqrt{11}, 9+2\sqrt{11})$

$f''(x) = \frac{22}{(x-3)^3}$

		3	
$f''(x)$	−	NOT defined	+
⇓			
$f(x)$	∩		∪

No point of inflection

vertical asymptote :

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} \frac{x^2+3x-7}{x-3} = -\infty$ and $\lim_{x\to3^+} f(x) = \lim_{x\to3^+} \frac{x^2+3x-7}{x-3} = +\infty$

$\therefore$ vertical asymptote : $x=3$.

oblique asymptote :

Note : $f(x) = \frac{x^2+3x-7}{x-3}$

By long division,

$$\begin{array}{r} x+6 \\ x-3\,\overline{)\,x^2+3x-7} \\ \underline{x^2-3x} \\ 6x-7 \\ \underline{6x-18} \\ 11 \end{array}$$

$x^2+3x-7 = (x-3)(x+6)+11$

$$f(x) = \frac{x^2+3x-7}{x-3} = \underbrace{x+6}_{\text{oblique asymptote}} + \frac{11}{x-3}$$

$\therefore$ oblique asymptote : $y = x+6$

Explanation : $\lim_{x\to\pm\infty} f(x) - (x+6) = \lim_{x\to\pm\infty} \frac{11}{x+3} = 0$

Remark . Using long division to find oblique asymptote only works for the case that $f(x)$ is a rational function , i.e. quotient of two polynomials.

In general, $m = \lim_{x\to+\infty} \frac{f(x)}{x}$,

$c = \lim_{x\to+\infty} f(x) - mx$.

If anyone of them does NOT exist, it means there is no obilque asymptote ,

if both limit exist, $y = mx+c$ is an oblique asymptote at positive infinity

(similar for negative infinity)

x-intercept : Solve $f(x) = 0$

$$\frac{x^2+3x-7}{x-3} = 0$$
$$x^2+3x-7 = 0$$
$$x = \frac{-3\pm\sqrt{37}}{2}$$

y-intercept : $f(0) = \frac{7}{3}$

Sketch $y=f(x)$.

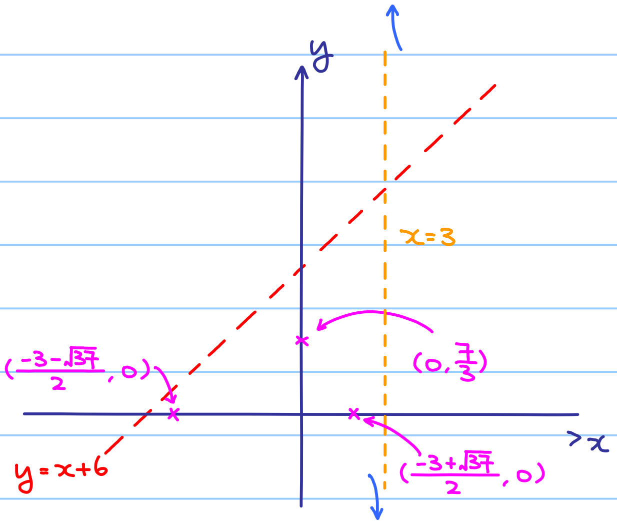
Step 1: draw asymptotes

Step 2: put down x-intercepts
and y-intercept

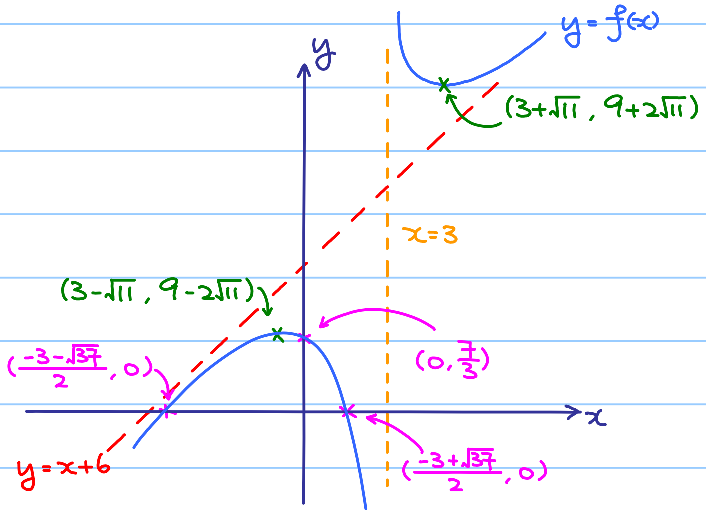
Step 3:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2+3x-7}{x-3} = -\infty$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2+3x-7}{x-3} = +\infty$$



Step 4: Use the information $f'(x)$ and $f''(x)$



Curve Sketching :

Goal : Given a function $f(x)$, sketch the graph of $y=f(x)$.

(Capturing main features)

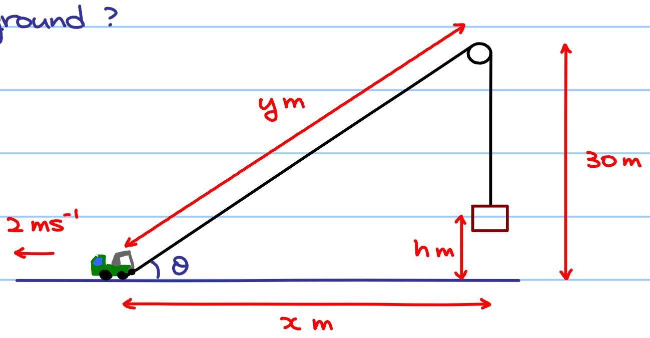
• x - intercept	solve $f(x)=0$
• y - intercept	y-intercept $= f(0)$
• increasing / decreasing	solve $f'(x)>0$ / $f'(x)<0$
saddle point / max. / min	change of sign of $f'(x)$?
• concave up / down	solve $f''(x)>0$ / $f''(x)<0$
point of inflection	change of sign of $f''(x)$?
• vertical asymptote	any $x=a$ with $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$
• horizontal asymptote	long division / $m = \lim_{x\to+\infty} \frac{f(x)}{x}$
• oblique asymptote	$c = \lim_{x\to+\infty} f(x) - mx$

6.7 Rate of Change

Recall : $\frac{dy}{dx}$ is the rate of change of y with respect to x.

Example 6.7.1

A weight is lifted by a rope which passes through a pulley. The other end of the rope is pulled by a truck that moves at $2\ ms^{-1}$. If the pulley is $30\,m$ above the ground, how fast is the weight rising when the rope makes an angle $\frac{\pi}{3}$ with the ground ?



Given : $\frac{dx}{dt} = 2$

Question : When $\theta = \frac{\pi}{6}$, $\frac{dh}{dt} = ?$

Relations of variables : $x^2 + 30^2 = y^2$ — (1)

$x \tan\theta = 30$ — (2)

$\frac{dy}{dt} = \frac{dh}{dt}$ — (3)

From (2), when $\theta = \frac{\pi}{6}$, $x \tan\frac{\pi}{6} = 30 \Rightarrow x = 30\sqrt{3}$

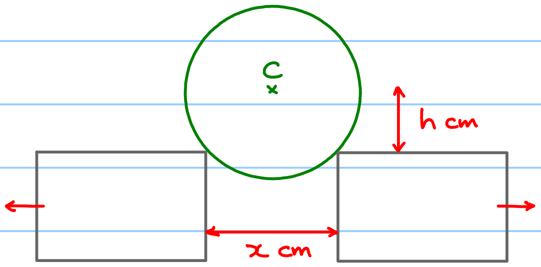
From (1), $(30\sqrt{3})^2 + 30^2 = y^2 \Rightarrow y = 60$

Differentiate (1) with respect to t. $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$2(30\sqrt{3})(2) = 2(60)\frac{dy}{dt}$

$\therefore \frac{dh}{dt} = \frac{dy}{dt} = \sqrt{3}$

Example 6.7.2



A cylinderical block fall vertically thrusting two rectangular blocks apart with equal horizontal velocities. When the center C of the cylinder is 20 cm above the level of the blocks, they are 30 cm apart and are moving at 10 cm s^{-1}. Find the velocity of the center of the cylinder

Question : When $x = 30$, $h = 20$, $\frac{dx}{dt} = 2(10) = 20$, $\frac{dh}{dt} = ?$

Relations of variables : $(\frac{x}{2})^2 + h^2 = R^2$ where R is the radius of the cylinder.

Differentiate with respect to t. $\frac{1}{2}x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$

When $x = 30$, $h = 20$, $\frac{dx}{dt} = 20$: $\frac{1}{2}(30)(20) + 2(20)\frac{dh}{dt} = 0$

$\frac{dh}{dt} = -\frac{15}{2}$

§ 7 Indeterminate Form and L'hôpital Rule

7.1 Indeterminate Form $\frac{0}{0}$

Consider $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ and suppose $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist.

Case 1. If $\lim\limits_{x\to a} g(x) \neq 0$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

Case 2: If $\lim\limits_{x\to a} g(x) = 0$ and $\lim\limits_{x\to a} f(x) \neq 0$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ does NOT exist. (e.g. $\lim\limits_{x\to 1}\frac{x}{x-1}$)

Case 3: If $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} f(x) = 0$, then we do NOT know whether $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ exist!

e.g. $\lim\limits_{x\to 0}\frac{x^2}{x} = \lim\limits_{x\to 0} x = 0$,

$\lim\limits_{x\to 0}\frac{x^2}{x^2} = \lim\limits_{x\to 0} 1 = 1$,

$\lim\limits_{x\to 0}\frac{x}{x^2} = \lim\limits_{x\to 0}\frac{1}{x}$ does not exist.

We call it indeterminate form $\frac{0}{0}$.

Theorem 7.1.1 (L'hôpital's Rule)

Suppose that $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, I is an open interval containing a, f and g are differentiable on $I\setminus\{a\}$, and $g'(x)\neq 0$ on $I\setminus\{a\}$.

If $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example 7.1.1

$\lim\limits_{x\to 0}\frac{\sin x}{x}$ $\left(\frac{0}{0}\right)$ — (*)

$= \lim\limits_{x\to 0}\frac{\cos x}{1}$ — (**) Logic: limit (**) exists ⇒ limit (*) exists

$= \frac{1}{1}$

$= 1$

Example 7.1.2

$\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$ $\left(\frac{0}{0}\right)$

$= \lim\limits_{x\to 0}\frac{\sin x}{2x}$ $\left(\frac{0}{0}\right)$

$= \lim\limits_{x\to 0}\frac{\cos x}{2}$

$= \frac{1}{2}$

Example 7.1.3

$$\lim_{x\to 0}\frac{e^x-1}{x}$$
$$=\lim_{x\to 0}\frac{e^x}{1}$$
$$=1$$

$$\lim_{x\to 0}\frac{e^{3x}-1}{2x} \quad \left(\frac{0}{0}\right)$$
$$=\lim_{x\to 0}\frac{e^{3x}}{2}$$
$$=1$$

(See theorem 3.6.1 and example 3.6.2)

7.2 Indeterminate Form $\frac{\infty}{\infty}$, $\infty\,0$, $\infty-\infty$

- L'hôpital's Rule can also be applied to $\frac{\infty}{\infty}$
- L'hôpital's Rule can also be applied to left hand limit or right hand limit

$$\lim_{x\to a^+}\frac{f(x)}{g(x)}=\lim_{x\to a^+}\frac{f'(x)}{g'(x)}, \quad \lim_{x\to a^-}\frac{f(x)}{g(x)}=\lim_{x\to a^-}\frac{f'(x)}{g'(x)}$$

- L'hôpital's Rule can also be applied to limits at infinities

$$\lim_{x\to +\infty}\frac{f(x)}{g(x)}=\lim_{x\to +\infty}\frac{f'(x)}{g'(x)}, \quad \lim_{x\to -\infty}\frac{f(x)}{g(x)}=\lim_{x\to -\infty}\frac{f'(x)}{g'(x)}$$

Example 7.2.1

$$\lim_{x\to \frac{\pi}{2}^-}\frac{\sec x}{1+\tan x} \quad \left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to \frac{\pi}{2}^-}\frac{\sec x\tan x}{\sec^2 x}$$

$$=\lim_{x\to \frac{\pi}{2}^-}\sin x$$

$$=1$$

Example 7.2.2

$$\lim_{x\to +\infty}\frac{\ln x}{2\sqrt{x}} \quad \left(\frac{\infty}{\infty}\right)$$
$$=\lim_{x\to +\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$$
$$=\lim_{x\to +\infty}\frac{1}{\sqrt{x}}$$
$$=0$$

Indeterminate Form $\infty \, 0$

Idea : Converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 7.2.3

$$\lim_{x \to +\infty} x \sin \frac{1}{x} \qquad (\infty \ 0)$$

↓ convert to

$$= \lim_{x \to +\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to +\infty} \frac{-\frac{1}{x^2} \cos \frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to +\infty} \cos \frac{1}{x}$$

$$= 1$$

Alternative method :

$$\lim_{x \to +\infty} x \sin \frac{1}{x} \qquad (\infty \ 0)$$

$$= \lim_{h \to 0^+} \frac{\sin h}{h} \qquad \left(\frac{0}{0}\right)$$

Let $h = \frac{1}{x}$,

As $x \to +\infty$, $h \to 0^+$

$$= 1$$

Example 7.2.4

$$\lim_{x \to 0^+} \sqrt{x} \ln x \qquad (\infty \ 0)$$

↓ convert to

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-\frac{3}{2}}}$$

$$= \lim_{x \to 0^+} -2\sqrt{x}$$

$$= 0$$

Remark: Why don't we try $\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\sqrt{x}}{\left(\frac{1}{\ln x}\right)}$ $\left(\frac{0}{0}\right)$?

Indeterminate Form $\infty - \infty$

Idea : Converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 7.2.5

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) \qquad (\infty - \infty)$$

$$= \lim_{x\to 0} \frac{x - \sin x}{x \sin x} \qquad \left(\frac{0}{0}\right) \quad \text{) convert to}$$

Ex : $\vdots$

$= 0$

7.3 Indeterminate Form 1^{∞}, 0^{0}, ∞^{0}

Indeterminate Form 1^{∞}, 0^{0}, ∞^{0}

Idea : Taking ln , converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 7.3.1

Find $\lim_{x\to 1^+} x^{\frac{1}{1-x}} \qquad (1^{\infty})$

Let $y = x^{\frac{1}{1-x}}$

$\ln y = \frac{\ln x}{1-x}$

$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{1-x} \qquad \left(\frac{0}{0}\right)$

$\ln\left(\lim_{x\to 1^+} y\right) = \lim_{x\to 1^+} \frac{\left(\frac{1}{x}\right)}{-1}$

$= -1$

$\ln\left(\lim_{x\to 1^+} x^{\frac{1}{1-x}}\right) = -1$

$\therefore \lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1}$

Example 7.3.2

Find $\lim_{x\to +\infty} x^{\frac{1}{x}} \qquad (\infty^{0})$

Let $y = x^{\frac{1}{x}}$

$\ln y = \frac{\ln x}{x}$

$\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty}\right)$

$\ln\left(\lim_{x\to +\infty} y\right) = \lim_{x\to +\infty} \frac{\left(\frac{1}{x}\right)}{1}$

$\ln\left(\lim_{x\to +\infty} x^{\frac{1}{x}}\right) = 0$

$\therefore \lim_{x\to +\infty} x^{\frac{1}{x}} = e^{0} = 1$

§ 8 Indefinite Integration

8.1 Antiderivatives

Definition 8.1.1

A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The process of finding antiderivatives is called indefinite integration.

Example 8.1.1

If $f(x) = 2x$, $F(x) = x^2$,

then we have $F'(x) = f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

However, consider $F(x) = x^2 + C$, where C is a constant.

Then, we still have $F'(x) = f(x)$.

Therefore, antiderivative of a function $f(x)$ is NOT unique.

That is why we call "an" antiderivative instead of "the" antiderivative.

Natural question : If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$,
what is the relation between them ?

Answer : $F(x)$ and $G(x)$ differ by a constant.

proof : Suppose $F'(x) = G'(x) = f(x)$

Let $H(x) = F(x) - G(x)$

Then $H'(x) = F'(x) - G'(x) = 0$

$\therefore$ $H(x)$ is a constant function, i.e. $H(x) = C$ for some constant C.

i.e $F(x) = G(x) + C$

(Refer to theorem 6.2.2)

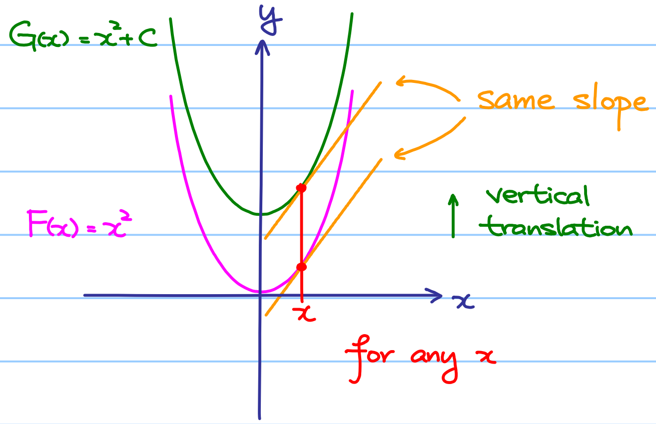
Therefore, antiderivative of a function $f(x)$ is NOT unique,
but it is unique up to a constant.

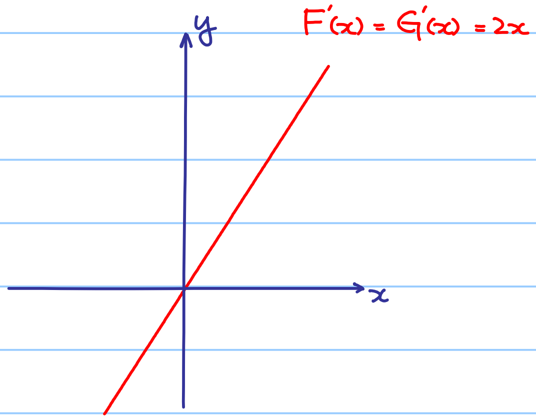
Example 8.1.2

If $f(x) = 2x$, $F(x) = x^2$

then we have $F'(x) = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

and all antiderivatives of $f(x)$ must be of the form $x^2 + C$.





If $F(x)$ is an antiderivative of $f(x)$, we write

$$\underbrace{\int}_{\text{integral symbol}} \overbrace{f(x)}^{\text{integrand}} \underbrace{dx}_{\text{variable of integration}} = F(x) + C$$

Example 8.1.2

$$\int 2x\, dx = x^2 + C$$

8.2 Rules of Indefinite Integration

Theorem 8.2.1

1) $\int k\,dx = kx + C$, for a constant k.

2) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, for all $n \neq -1$.

3) $\int \frac{1}{x}\,dx = \ln|x| + C$

4) $\int e^x\,dx = e^x + C$

5) $\int \cos x\,dx = \sin x + C$

6) $\int \sin x\,dx = -\cos x + C$

7) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

proof:

Derivative of RHS = Integrand on LHS

Theorem 8.2.2

1) $\int kf(x)\,dx = k\int f(x)\,dx$

2) $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

proof:

1) $\frac{d}{dx}\left(\int kf(x)\,dx\right) = \frac{d}{dx}\left(k\int f(x)\,dx\right) = kf(x)$

2) $\frac{d}{dx}\left(\int f(x) \pm g(x)\,dx\right) = \frac{d}{dx}\left(\int f(x)\,dx \pm \int g(x)\,dx\right) = f(x) \pm g(x)$

Example 8.2.1

$\int 2x^5 - 3x^2 + 7x + 5\,dx$

$\int dx$ means $\int 1\,dx$

$= 2\int x^5\,dx - 3\int x^2\,dx + 7\int x\,dx + 5\int dx$

$\int$ still there,

No need to add $+C$!

$= 2\left(\frac{x^6}{6}\right) - 3\left(\frac{x^3}{3}\right) + 7\left(\frac{x^2}{2}\right) + 5x + C$

$= \frac{x^6}{3} - x^3 + \frac{7x^2}{2} - 5x + C$

Example 8.2.2

$\int \frac{x^3 - 5}{x}\,dx$

$= \int x^2 - \frac{5}{x}\,dx$

$= \frac{x^3}{3} - 5\ln|x| + C$

Example 8.2.3

Find a function $F(x)$ such that $F(0) = 3$ and $F'(x) = 2x$.

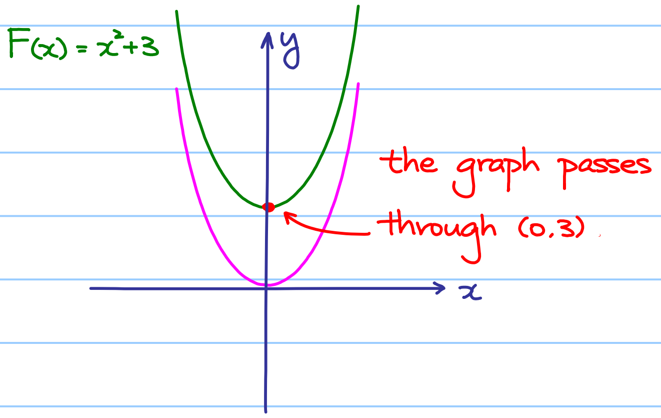
$F'(x) = 2x$

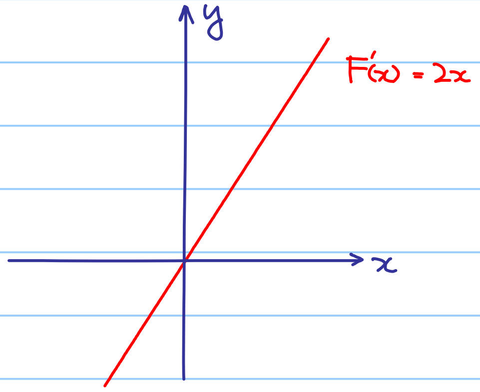
$F(x) = \int 2x \, dx$

$\quad = x^2 + C$

$F(0) = 0^2 + C = 3 \quad \Rightarrow \quad C = 3$

$\therefore F(x) = x^2 + 3$





8.3 Integration by Substitution

Question : $\int (2x+1)^{10} \, dx = ?$

Hard to integrate by expanding the polynomial.

Solution : Integration by Substitution

Theorem 8.3.1

$$\int f(u(x)) \, u'(x) \, dx = \int f(u) \, du \qquad \text{OR : } \int f(x) \frac{du}{dx} dx = \int f(u) \, du$$

proof :

$$\frac{d}{dx} \int f(u(x)) \, u'(x) \, dx = f(u(x)) \, u'(x)$$

$$\frac{d}{dx} \int f(u) \, du = \frac{d}{du} \int f(u) \, du \; \frac{du}{dx} \qquad \text{(Chain Rule)}$$

$$= f(u(x)) \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \int f(u(x)) \, u'(x) \, dx = \frac{d}{dx} \int f(u) \, du$$

$$\therefore \int f(u(x)) \, u'(x) \, dx = \int f(u) \, du$$

Example 8.3.1

$\int (2x+1)^{10}\,dx = ?$

Let $u(x) = 2x+1$ $\quad u'(x) = 2$

$f(u) = u^{10}$ $\quad f(u(x)) = (2x+1)^{10}$

$$\int (2x+1)^{10}\,dx = \frac{1}{2}\int \underset{\underset{f(u(x))}{\uparrow}}{(2x+1)^{10}} \cdot \underset{\underset{u'(x)}{\uparrow}}{2}\,dx = \frac{1}{2}\int \underset{\underset{f(u)}{\uparrow}}{u^{10}}\,du$$

$$= \frac{1}{22}u^{11} + C = \frac{1}{22}(2x+1)^{11} + C$$

But, usually we write,

$\int (2x+1)^{10}\,dx$

$= \int u^{10} \cdot \frac{1}{2}du$

$= \frac{1}{22}u^{11} + C$

$= \frac{1}{22}(2x+1)^{11} + C$

Let $u = 2x+1$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}du$

(called differential form, can be defined rigorously)

Example 8.3.2

$\int e^{ax}\,dx$

$= \int e^{u} \cdot \frac{1}{a}du$

$= \frac{1}{a}e^{u} + C$

$= \frac{1}{a}e^{ax} + C$

Let $u = ax$

$\frac{du}{dx} = a$

$dx = \frac{1}{a}du$

Example 8.3.3

$\int 6x(4x^2+3)^7\,dx$

$= \int 6(4x^2+3)^7\,x\,dx$

$= \int 6u^7 \frac{1}{8}dx$

$= \frac{6}{8} \cdot \frac{1}{8}u^8 + C$

$= \frac{3}{32}(4x^2+3)^8 + C$

Let $u = 4x^2+3$

$\frac{du}{dx} = 8x$

$x\,dx = \frac{1}{8}du$

Example 8.3.4

$\int \frac{(\ln x)^2}{x} dx \quad , \quad x > 0$

$\int \frac{(\ln x)^2}{x} dx$ Let $u = \ln x$

$= \int u^2 \, du$ $\frac{du}{dx} = \frac{1}{x}$

$= \frac{1}{3} u^3 + C$ $\frac{1}{x} dx = du$

$= \frac{1}{3} (\ln x)^3 + C$

Question : How to make a guess of $u(x)$?

Integration by Substitution : $\int f(u(x)) \, u'(x) \, dx = \int f(u) \, du$

Example : $\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \cdot \frac{1}{x} dx$ Let $u = \ln x$

Realize the integrand as a product of parts and make a guess of $u(x)$ such that one part can be realized as a function $f(u)$, another part is $u'(x)$

Exercise 8.3 1

1) Show that $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$ Hint : Let $u = ax+b$

2) Evaluate

a) $\int x^3 e^{x^4} dx$ Hint : Let $u = x^4$ Ans : $\frac{1}{4} e^{x^4} + C$

b) $\int 6x \sqrt{x^2+3} \, dx$ Hint : Let $u = x^2+3$ Ans : $2 (x^2+3)^{\frac{3}{2}} + C$

Integration of Expontenial Functions :

Recall : $\int e^{kx} dx = \frac{1}{k} e^{kx} + C$

In general : $\int a^x dx = ?$ for $a > 0$

Recall : $a^x = e^{\ln a^x} = e^{(\ln a)x}$

$\therefore \int a^x dx = \int e^{(\ln a)x} dx$

$= \frac{1}{\ln a} e^{(\ln a)x} + C$

$= \frac{a^x}{\ln a} + C$

OR : Recall that $\frac{d}{dx} a^x = a^x \ln a$

so $\frac{d}{dx} \frac{a^x}{\ln a} = a^x$, and $\int a^x dx = \frac{a^x}{\ln a} + C$

Integration of Logarithmic Functions :

$\int \ln x \, dx = ?$ for $x > 0$

Exercise : $\frac{d}{dx} x\ln x - x$

Ans : $\ln x$!

Therefore , $\int \ln x \, dx = x\ln x - x + C$

Problem : How do we know $\frac{d}{dx} x\ln x - x = \ln x$ in advance ?

(Make a guess of antiderivative of $\ln x$ directly)

Any direct way to find an antiderivative of $\ln x$? (Yes, later !)

Example 8.3.5 (Constant issue)

$\int (x+1)^2 \, dx$ let $u = x+1$

$= \int u^2 \, du$ $du = dx$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}(x+1)^3 + C$

$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + C$

$\int (x+1)^2 \, dx$

$= \int x^2 + 2x + 1 \, dx$

$= \frac{1}{3}x^3 + x^2 + x + C$

seems to be different !

Ans : This C is NOT that C !

Integration of Rational Functions :

Rational Functions : a quotient of two polynomials

Rational Function $\longrightarrow$ $R(x) = \dfrac{p(x)}{q(x)}$ $\leftarrow\!\!>$ polynomials

Simplest case : $\deg q(x) = 1$ i.e. $q(x) = ax+b$ where $a \neq 0$.

- $\displaystyle\int \frac{p(x)}{ax+b}\,dx$

By long division, $p(x) = (ax+b)\,u(x) + R$

$$\frac{p(x)}{ax+b} = u(x) + \frac{R}{ax+b}$$

$$\begin{array}{r} u(x) \\ ax+b \,\overline{)\, p(x)} \\ \vdots \\ \hline R \end{array}$$

Then $\displaystyle\int \frac{p(x)}{ax+b}\,dx = \int u(x) + \frac{R}{ax+b}\,dx$

We know how to integrate !

Example 8.3.6

$$\int \frac{x^2+3x+5}{x+1}\,dx$$

$$= \int x+2+\frac{3}{x+1}\,dx$$

$$= \frac{x^2}{2} + 2x + 3\ln|x+1| + C$$

$$\begin{array}{r} x+2 \\ x+1 \,\overline{)\, x^2+3x+5} \\ \underline{x^2+x} \\ 2x+5 \\ \underline{2x+2} \\ 3 \end{array}$$

$\therefore\ x^2+3x+5 = (x+1)(x+2)+3$

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}$$

Exercise : Evaluate $\displaystyle\int \frac{6x^2-5x+1}{3x-2}\,dx$

Ans : $x^2 - \frac{x}{3} + \frac{1}{9}\ln|3x-2| + C$

Next case : $\deg q(x) = 2$ i.e. $q(x) = ax^2+bx+c$ where $a \neq 0$.

If $\deg p(x) \geq 2$, by long division, $\displaystyle\int \frac{p(x)}{ax^2+bx+c}\,dx = \int u(x) + \frac{rx+s}{ax^2+bx+c}\,dx$

$\uparrow$ polynomial

$$\begin{array}{r} u(x) \\ ax^2+bx+c \,\overline{)\, p(x)} \\ \hline rx+s \end{array}$$

Just focus on $\displaystyle\int \frac{rx+s}{ax^2+bx+c}\,dx$

Recall : $\Delta = b^2 - 4ac$

We further consider 3 subcases :

(i) $\Delta > 0$ (ii) $\Delta = 0$ (iii) $\Delta < 0$

(i) $\Delta > 0$, $q(x) = ax^2+bx+c = (m_1x+n_1)(m_2x+n_2)$

Express $\frac{rx+s}{ax^2+bx+c}$ into the form $\frac{A}{m_1x+n_1}+\frac{B}{m_2x+n_2}$.

Then $\int \frac{rx+s}{ax^2+bx+c}\,dx = \int \frac{A}{m_1x+n_1}+\frac{B}{m_2x+n_2}\,dx$

We know how to integrate!

Example 8.3.7

$$\int \frac{5x-7}{x^2-2x-3}\,dx$$

Note : $\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)}$

Suppose $\frac{5x-7}{(x-3)(x+1)} \equiv \frac{A}{x-3}+\frac{B}{x+1}$

$\Rightarrow 5x-7 \equiv A(x+1)+B(x-3)$

$\Rightarrow A=2 , B=3 .$

$$\int \frac{5x-7}{x^2-2x-3}\,dx = \int \frac{2}{x-3}+\frac{3}{x+1}\,dx = 2\ln|x-3|+3\ln|x+1|+C$$

Exercise : Evaluate $\int \frac{40}{x(200-x)}\,dx$

Ans : $\frac{1}{5}(\ln|x|-\ln|200-x|)+C = \frac{1}{5}\ln\left|\frac{x}{200-x}\right|+C$

(ii) $\Delta = 0$, $q(x) = ax^2+bx+c = (mx+n)^2$

Express $\frac{rx+s}{ax^2+bx+c}$ into the form $\frac{A}{(mx+n)^2}+\frac{B}{mx+n}$.

Then $\int \frac{rx+s}{ax^2+bx+c}\,dx = \int \frac{A}{(mx+n)^2}+\frac{B}{mx+n}\,dx$

We know how to integrate!

Example 8.3.8

$$\int \frac{2x-1}{(x-2)^2}\,dx$$

Suppose $\frac{2x-1}{(x-2)^2} \equiv \frac{A}{(x-2)^2}+\frac{B}{x-2}$

$\Rightarrow 2x-1 \equiv A+B(x-2)$

$\Rightarrow A=3 , B=2$

$$\int \frac{2x-1}{(x-2)^2}\,dx = \int \frac{3}{(x-2)^2}+\frac{2}{x-2}\,dx = \frac{-3}{x-2}+2\ln|x-2|+C$$

Exercise : Evaluate $\int \frac{4x+2}{(2x-1)^2}\,dx$

Ans : $\frac{-2}{2x-1}+\ln|2x-1|+C$

(iii) $\Delta < 0$, $q(x) = ax^2+bx+c$ cannot be factorized as a product of two linear factors

$\int \frac{1}{x^2+a^2}\,dx$ let $x = au$

$= \int \frac{1}{a^2u^2+a^2}\,a\,du$ $dx = a\,du$

$= \frac{1}{a}\int \frac{1}{u^2+1}\,du$

$= \frac{1}{a}\tan^{-1}u + C$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Example 8.3.9

$\int \frac{1}{x^2+2x+5}\,dx$

$= \int \frac{1}{u^2+2^2}\,du$ let $u = x+1$ (or let $x+1 = 2t$, what happens ?)

$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$ $du = dx$

$= \frac{1}{2}\tan^{-1}\frac{x+1}{2} + C$

Example 8.3.10

$\int \frac{4x+7}{x^2+2x+5}\,dx$ Note : $d(x^2+2x+5) = (2x+2)\,dx$

$= \int \frac{2(2x+2)+3}{x^2+2x+5}\,dx$ and $4x+7 = 2(2x+2)+3$

$= 2\int \frac{2x+2}{x^2+2x+5}\,dx + 3\int \frac{1}{x^2+2x+5}\,dx$

$= 2\ln(x^2+2x+5) + 3\left(\frac{1}{2}\tan^{-1}\frac{x+1}{2}\right) + C$

$= 2\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\frac{x+1}{2} + C$

General case : $\deg q(x) > 2$

Partial fraction : resolve $\frac{p(x)}{q(x)}$ into a sum of simpler fractions.

Then , it reduces to the above cases

Integration of Trigonometric Functions :

- $\int \tan x \, dx$ and $\int \cot x \, dx$

$$\int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} \, dx$$ let $u = \cos x$

$$= \int -\frac{1}{u} \, du$$ $\frac{du}{dx} = -\sin x$

$$= -\ln|u| + C$$ $-du = \sin x \, dx$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

$$\int \cot x \, dx$$

$$= \int \frac{\cos x}{\sin x} \, dx$$ let $u = \sin x$

Ex : ⋮

$$= \ln|\sin x| + C$$

- $\int \sec x \, dx$ and $\int \csc x \, dx$, t-formula

t-formula :

Let $t = \tan \frac{x}{2}$

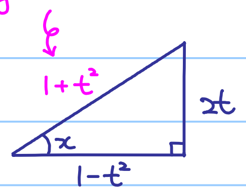
Idea : We can express all trigonometric functions in terms of t.

Note : $\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$ and so $\cot x = \frac{1-t^2}{2t}$

$\therefore \sin x = \frac{2t}{1+t^2}$ and so $\csc x = \frac{1+t^2}{2t}$

$\cos x = \frac{1-t^2}{1+t^2}$ $\sec x = \frac{1+t^2}{1-t^2}$

By Pyth. thm.



Therefore, all trigonometric functions in terms of t.

Note: $t = \tan \frac{x}{2}$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1+t^2)$$

$$dx = \frac{2}{1+t^2} \, dt$$

Idea : $\int f(\sin x, \cos x) \, dx$

$$= \int \underbrace{f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2}}_{\text{Rational functions of } t.} \, dt$$

Transforming an integral of trigonometric function to an integral of rational function.

$\int \csc x \, dx$

$= \int \frac{1+t^2}{2t} \frac{2}{1+t^2} dt$

$= \int \frac{1}{t} dt$

$= \ln|t| + C$

$= \ln\left|\tan\frac{x}{2}\right| + C$

$\int \sec x \, dx$

$= \int \frac{1+t^2}{1-t^2} \frac{2}{1+t^2} dt$

$= \int \frac{2}{1-t^2} dt$

$= \int \frac{1}{1+t} + \frac{1}{1-t} \, dt$

$= \ln|1+t| - \ln|1-t| + C$

$= \ln\left|\frac{1+t}{1-t}\right| + C$

$= \ln\left|\frac{2t+1-t^2}{1-t^2}\right| + C$

$= \ln|\tan x + \sec x| + C$

Example 8.3.11

$\int \frac{1}{1+\cos x} dx$

$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt$

$= \int dt$

$= t + C$

$= \tan\frac{x}{2} + C$

Exercise 8.3 2

Show that

a) $\int \sin px \, dx = -\frac{1}{p}\cos px + C$

b) $\int \cos px \, dx = \frac{1}{p}\sin px + C$

- $\int \sin px \cos qx \, dx$, $\int \sin px \sin qx \, dx$, $\int \cos px \cos qx \, dx$

Recall : $\sin px \cos qx = \frac{1}{2}[\sin(p+q)x + \sin(p-q)x]$

$\cos px \cos qx = \frac{1}{2}[\cos(p+q)x + \cos(p-q)x]$

$\sin px \sin qx = -\frac{1}{2}[\cos(p+q)x - \cos(p-q)x]$

We know how to integrate RHS !

Example 8.3.12

$\int \sin 5x \cos 3x \, dx$

$= \frac{1}{2}\int \sin 8x + \sin 2x \, dx$

$= \frac{1}{2}(-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}) + C$

$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$

In particular, $\cos^2 px = \frac{1}{2}(1+\cos 2px)$

$\sin^2 px = \frac{1}{2}(1-\cos 2px)$

Example 8.3.13

$\int \cos x \cos^2 3x \, dx$

$= \int \cos x \, [\frac{1}{2}(1+\cos 6x)] \, dx$

$= \frac{1}{2}\int \cos x \, dx + \frac{1}{2}\int \cos x \cos 6x \, dx$

$= \frac{1}{2}\int \cos x \, dx + \frac{1}{4}\int \cos 7x + \cos 5x \, dx$

$= \frac{\sin x}{2} + \frac{\sin 7x}{28} + \frac{\sin 5x}{10} + C$

Exercise : Find $\int \sin x \sin 3x \sin 6x \, dx$

Ans : $\frac{\cos 10x}{40} + \frac{\cos 2x}{8} - \frac{\cos 8x}{10} - \frac{\cos 4x}{16} + C$

Exercise 8.3 2

Show that

a) $\int \sin px \, dx = -\frac{1}{p} \cos px + C$

b) $\int \cos px \, dx = \frac{1}{p} \sin px + C$

- $\int \sin px \cos qx \, dx$, $\int \sin px \sin qx \, dx$, $\int \cos px \cos qx \, dx$

Recall : $\sin px \cos qx = \frac{1}{2}[\sin(p+q)x + \sin(p-q)x]$

$\cos px \cos qx = \frac{1}{2}[\cos(p+q)x + \cos(p-q)x]$

$\sin px \sin qx = -\frac{1}{2}[\cos(p+q)x - \cos(p-q)x]$

We know how to integrate RHS !

Example 8.3.12

$\int \sin 5x \cos 3x \, dx$

$= \frac{1}{2} \int \sin 8x + \sin 2x \, dx$

$= \frac{1}{2} \left(-\frac{\cos 8x}{8} - \frac{\cos 2x}{2} \right) + C$

$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$

In particular, $\cos^2 px = \frac{1}{2}(1 + \cos 2px)$

$\sin^2 px = \frac{1}{2}(1 - \cos 2px)$

Example 8.3.13

$\int \cos x \cos^2 3x \, dx$

$= \int \cos x \left[\frac{1}{2}(1 + \cos 6x)\right] dx$

$= \frac{1}{2} \int \cos x \, dx + \frac{1}{2} \int \cos x \cos 6x \, dx$

$= \frac{1}{2} \int \cos x \, dx + \frac{1}{4} \int \cos 7x + \cos 5x \, dx$

$= \frac{\sin x}{2} + \frac{\sin 7x}{28} + \frac{\sin 5x}{20} + C$

Exercise : Find $\int \sin x \sin 3x \sin 6x \, dx$

Ans : $\frac{\cos 10x}{40} + \frac{\cos 2x}{8} - \frac{\cos 8x}{32} - \frac{\cos 4x}{16} + C$

- $\int \sin^m x \cos^n x \, dx$

Case 1 : m is odd

Apply : $\sin x \, dx = -d\cos x$ and $\sin^2 x = 1-\cos^2 x$

Example 8.3.14

$\int \sin^3 x \cos^2 x \, dx$

$= \int \sin^2 x \sin x \cos^2 x \, dx$

$= -\int \sin^2 x \cos^2 x \, d\cos x$

$= -\int (1-\cos^2 x)\cos^2 x \, d\cos x$

$= \int -\cos^2 x + \cos^4 x \, d\cos x$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

Case 2 : n is odd

Similar to case 1

Apply : $\cos x \, dx = d\sin x$ and $\cos^2 x = 1-\sin^2 x$

Example 8.3.15

$\int \sin^4 x \cos^3 x \, dx$

$= \int \sin^4 x \cos^2 x \cos x \, dx$

$= \int \sin^4 x (1-\sin^2 x) \, d\sin x$

⋮ Ex

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

Case 3 : m and n are even.

Apply : $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin x \cos x = \frac{1}{2}\sin 2x$

Example 8.3.16

$\int \sin^2 x \cos^4 x \, dx$

$= \int (\sin x \cos x)^2 \cos^2 x \, dx$

$= \int (\frac{1}{4}\sin^2 2x)(\frac{1+\cos 2x}{2}) \, dx$

$= \frac{1}{8}\underbrace{\int \sin^2 2x \, dx}_{\text{case 3 again}} + \frac{1}{8}\underbrace{\int \sin^2 2x \cos 2x \, dx}_{\text{reduce to case 1}}$

$= \frac{1}{16}\int 1-\cos 4x \, dx + \frac{1}{8}\int \sin^2 2x \, \frac{1}{2} d\sin 2x$

$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$

- $\int \tan^m x \sec^n x \, dx$

Case 1. m is odd

Apply : $\tan x \sec x \, dx = d \sec x$ and $\tan^2 x = 1 - \sec^2 x$

Example 8.3.17

$\int \tan^3 x \sec^4 x \, dx$

$= \int \tan^2 x \tan x \sec^3 x \sec x \, dx$

$= \int \tan^2 x \sec^3 x \, d\sec x$

$= \int (\sec^2 x - 1) \sec^3 x \, d\sec x$

$= \int \sec^5 x - \sec^3 x \, d\sec x$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

Case 2. n is even

Similar to case 1

Apply : $\sec^2 x \, dx = d\tan x$ and $\sec^2 x = 1 + \tan^2 x$

Example 8.3.18

$\int \tan^4 x \sec^4 x \, dx$

$= \int \tan^4 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^4 x (1 + \tan^2 x) \, d\tan x$

⋮ Ex

$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

Case 3. m is even and n is odd

Using integration by parts, later!

- $\int \csc^m x \cot^n x \, dx$

Similarly, apply $\csc^2 x = -d\cot x$

$\csc x \cot x = -d\csc x$

$1 + \cot^2 x = \csc^2 x$

Exercise : Find

a) $\int \csc^6 x \cot^4 x \, dx$ Ans : $-\frac{\cot^9 x}{9} - \frac{2\cot^7 x}{7} - \frac{\cot^5 x}{5} + C$

b) $\int \csc^5 x \cot^3 x \, dx$ $-\frac{\csc^7 x}{7} + \frac{\csc^5 x}{5} + C$

Integration of Irrational Functions :

- Integrand with $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ $(a>0)$

(1) For $\sqrt{a^2-x^2}$, we let $x=a\sin\theta$ $\quad -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$

(2) For $\sqrt{a^2+x^2}$, we let $x=a\tan\theta$ $\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

(3) For $\sqrt{x^2-a^2}$, we let $x=a\sec\theta$ $\quad 0\leqslant\theta\leqslant\pi$

Example 8.3.19

$\int x^3\sqrt{4-x^2}\,dx$ $\qquad$ Let $x=2\sin\theta$

$=\int 8\sin^3\theta\sqrt{4\cos^2\theta}\,(2\cos\theta)\,d\theta$ $\qquad$ $dx=2\cos\theta d\theta$

$=\int 32\cos^2\theta\sin^3\theta\,d\theta$

$=\int 32\cos^2\theta\sin^2\theta\sin\theta\,d\theta$ $\qquad$ $x=2\sin\theta \Rightarrow \sin\theta=\frac{x}{2}$

$=\int 32\cos^2\theta(1-\cos^2\theta)\,d(-\cos\theta)$ $\qquad$ $\cos\theta=\pm\sqrt{1-\sin^2\theta}=\sqrt{1-(\frac{x}{2})^2}=\pm\frac{\sqrt{4-x^2}}{2}$

$=\int 32\cos^4\theta-32\cos\theta\,d\cos\theta$ $\qquad$ $-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}\Rightarrow\cos\theta>0$

$=\frac{32}{5}\cos^5\theta-\frac{32}{3}\cos^3\theta+C$ $\qquad$ $\therefore\cos\theta=\frac{\sqrt{4-x^2}}{2}$

$=\frac{32}{5}(\frac{\sqrt{4-x^2}}{2})^5-\frac{32}{3}(\frac{\sqrt{4-x^2}}{2})^3+C$

$=-\frac{1}{15}(3x^2+8)(4-x^2)^{\frac{3}{2}}+C$

Note : $\sqrt{a^2-x^2}$ is well-defined only when $a^2-x^2\geqslant 0$, that means $-a<x<a$

Also we have $-1\leqslant\sin\theta\leqslant 1$ when $-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$,

so $-a\leqslant a\sin\theta\leqslant a$, that is the reason why we let $x=a\sin\theta$.

Think : How about $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$?

Example 8.3.20

$\int\frac{\sqrt{x^2-4}}{x^3}\,dx$ $\qquad$ Let $x=2\sec\theta$

$=\int\frac{\sqrt{4\tan^2\theta}}{8\sec^3\theta}\,2\sec\theta\tan\theta\,d\theta$ $\qquad$ $dx=2\sec\theta\tan\theta\,d\theta$

$=\frac{1}{2}\int\sin^2\theta\,d\theta$

$=\frac{1}{4}\int 1-\cos 2\theta\,d\theta$

$=-\frac{1}{8}\sin 2\theta+\frac{\theta}{4}+C$

⋮ Ex

$=-\frac{\sqrt{x^2-4}}{2x^2}+\frac{1}{4}\cos^{-1}\frac{2}{x}+C$

Exercise 8.3 3

Show that, for $a>0$,

a) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) + C$

b) $\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| + C$

8.4 Integration by Parts

Recall: Let $u(x)$ and $v(x)$ be differentiable functions.

Product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrate both sides with respect to x:

$$\int u\frac{dv}{dx}\,dx = \int\frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

OR: $\int u\,dv = uv - \int v\,du$

Integration by Parts. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Example 8.4.1

$$\int x^2\ln x\,dx = \int(\ln x)\,x^2\,dx$$

$= \int(\ln x)\frac{d}{dx}\left(\frac{x^3}{3}\right)dx$ (Now, $u=\ln x$, $v=\frac{x^3}{3}$)

$= \int \ln x\; d\frac{x^3}{3}$

$= \frac{x^3}{3}\ln x - \int\frac{x^3}{3}\,d(\ln x)$

$= \frac{x^3}{3}\ln x - \int\frac{x^3}{3}\frac{1}{x}\,dx$

$= \frac{x^3}{3}\ln x - \int\frac{x^2}{3}\,dx$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$ (Verify the answer by differentiation!)

Example 8.4.2

$\int x e^x dx$ Note : $\frac{d}{dx} e^x = e^x$

$= \int x \, de^x$ $e^x dx = de^x$

$= xe^x - \int e^x dx$ Now, $u = x$, $v = e^x$

$= xe^x - e^x + C$

$= e^x(x-1) + C$

Remark: Why don't we try the following?

$\int xe^x dx$

$= \int e^x x dx$

$= \int e^x d(\frac{x^2}{2})$

$\vdots$

What happens?

Example 8.4.3

$\int x^2 e^x dx$

$= \int x^2 de^x$

$= x^2 e^x - \int e^x dx^2$

$= x^2 e^x - \underbrace{\int 2xe^x dx}$

Ex : Apply Integration by parts again!

Ans : $e^x(x^2 - 2x + 2) + C$

Question : How to make a guess of $u(x)$ and $v(x)$?

Integration by Parts : $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Example : $\int x^2 \ln x \, dx = \int (\ln x) \, x^2 \, dx$

$= \int (\ln x) \frac{d}{dx}(\frac{x^3}{3}) \, dx$

Realize the integrand as a product of parts and make a guess of $u(x)$ and $v(x)$ such that one part can be realized as a function $u(x)$, another part is $v'(x)$

Example 8.4.4

$\int x \sin 3x \, dx$

$= \int x \, d(-\frac{1}{3}\cos 3x)$

$= -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x \, dx$

$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$

Integration of Logarithmic Functions :

$\int \ln x \, dx = ?$ for $x > 0$

Using Integration by part :

$\int \ln x \, dx = x \ln x - \int x \, d \ln x$ $u = \ln x$ $v = x$

$= x \ln x - \int x \cdot \frac{1}{x} dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

Exercise 8.4.1

Find $\int \log_a x \, dx$

Hints : $\log_a x = \frac{\ln x}{\ln a}$

$\int \log_a x \, dx = \frac{1}{\ln a} \int \ln x \, dx$

$= \frac{1}{\ln a}(x \ln x - x + C)$

$= x\frac{\ln x}{\ln a} - \frac{x}{\ln a} + \frac{C}{\ln a}$

$= x\log_a x - \frac{x}{\ln a} + C'$ $C' = \frac{C}{\ln a}$ just a constant !

Example 8.4.5 (Transformed into the original Integral)

$$\int e^x \cos x\,dx = \int e^x\,d\sin x$$

$$= e^x \sin x - \int \sin x\,de^x$$

$$= e^x \sin x - \int e^x \sin x\,dx$$

$$= e^x \sin x - \int e^x\,d(-\cos x)$$

$$= e^x \sin x - (-e^x \cos x - \int -\cos x\,de^x)$$

$$= e^x \sin x - (-e^x \cos x - \int - e^x \cos x\,dx)$$

$$= e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x\,dx}_{\text{back to itself !}}$$

Be careful of +/- !

$$\therefore\ 2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C'$$ ← Don't forget !

$$\int e^x \cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) + C \qquad (C = \tfrac{1}{2}C')$$

Example 8.4.6

$$\int \sin(\ln x)\,dx$$

$$= x\sin(\ln x) - \int x\,d\sin(\ln x)$$

$$= x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$$= x\sin(\ln x) - (x\cos(\ln x) - \int x\,d\cos(\ln x))$$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx$$

$$\therefore \int \sin(\ln x)\,dx = \frac{1}{2}x[\sin(\ln x) + \cos(\ln x)] + C$$

Exercise 8.4.2

$$\int \sec^3 x\,dx$$

$$= \int \sec x\,(\sec^2 x)\,dx$$

$$= \int \sec x\,d\tan x$$

Ex :

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\therefore \int \sec^3 x\,dx = \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] + C$$

Think : In general, how to find $\int \tan^m x \sec^n x\,dx$ if m is even, n is odd ?

8.5 Reduction Formulae

Idea : Obtain a formula to reduce the complexity of the integrand.

Example 8.5.1

Let $I_n = \int x^n e^x dx$, where n is a nonnegative integer.

Prove that $I_n = x^n e^x - n I_{n-1}$, for $n \geq 1$.

$$\begin{aligned} I_n &= \int x^n e^x dx \\ &= \int x^n de^x \\ &= x^n e^x - \int e^x dx^n \\ &= x^n e^x - \int n e^x x^{n-1} dx \\ &= x^n e^x - n I_{n-1} \end{aligned}$$

Note : $I_0 = \int e^x dx = e^x + C$

We can apply this formula repeatly until we see I_0 :

$$\begin{aligned} \int x^3 e^x dx = I_3 &= x^3 e^x - 3 I_2 \\ &= x^3 e^x - 3(x^2 e^x - 2 I_1) \\ &= x^3 e^x - 3(x^2 e^x - 2(xe^x - 1 \cdot I_0)) \\ &= x^3 e^x - 3x^2 e^x + 3 \cdot 2 x e^x - 3 \cdot 2 \cdot 1 \cdot I_0 \\ &= x^3 e^x - 3x^2 e^x + 3 \cdot 2 x e^x - 3 \cdot 2 \cdot 1 \cdot e^x + C \\ &= x^3 e^x - P_1^3 x^2 e^x + P_2^3 x e^x - P_3^3 e^x + C \\ &= \left[\sum_{r=0}^{3} (-1)^r P_r^3 x^{3-r} e^x \right] + C \end{aligned}$$

In general, $\int x^n e^x dx = \left[\sum_{r=0}^{n} (-1)^r P_r^n x^{n-r} e^x \right] + C$ for $n \geq 1$.

The formula $I_n = x^n e^x - n I_{n-1}$ is called a reduction formula.

Example 8.5.2

Let $I_n = \int \tan^n x \, dx$, where n is a nonnegative integer.

Show that $I_n = \underbrace{\frac{1}{n-1} \tan^{n-1} x}_{} - I_{n-2}$ for $n \geq 2$.

Why / How do we get this?

$\int \tan^{n-2} x \; d \tan x$

$$
\begin{aligned}
I_n &= \int \tan^n x \, dx \\
&= \int \tan^{n-2} x \; \tan^2 x \; dx \\
&= \int \tan^{n-2} x \; (\sec^2 x - 1) \, dx \\
&= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx \\
&= \int \tan^{n-2} x \; d \tan x - I_{n-2} \\
&= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}
\end{aligned}
$$

As we can see, the index n is decreased by 2 when the reduction formula is applied, so we have two cases:

Case 1: start from an even integer $n = 2m$

$$
\begin{aligned}
I_{2m} &= \frac{1}{2m-1} \tan^{2m-1} x + I_{2m-2} \\
&= \frac{1}{2m-1} \tan^{2m-1} x + \frac{1}{2m-3} \tan^{2m-3} x + I_{2m-4} \\
&\;\vdots \\
&= \frac{1}{2m-1} \tan^{2m-1} x + \frac{1}{2m-3} \tan^{2m-3} x + \cdots + \frac{1}{3} \tan^3 x + \tan x + I_0 \qquad \text{(end at } I_0 \text{)} \\
&= \frac{1}{2m-1} \tan^{2m-1} x + \frac{1}{2m-3} \tan^{2m-3} x + \cdots + \frac{1}{3} \tan^3 x + \tan x + x + C \qquad \left(I_0 = \int dx = x + C\right)
\end{aligned}
$$

Case 2: start from an odd integer $n = 2m+1$

$$
\begin{aligned}
I_{2m+1} &= \frac{1}{2m} \tan^{2m} x + I_{2m-1} \\
&\;\vdots \\
&= \frac{1}{2m} \tan^{2m} x + \frac{1}{2m-2} \tan^{2m-2} x + \cdots + \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x + I_1 \qquad \text{(end at } I_1 \text{)} \\
&= \frac{1}{2m} \tan^{2m} x + \frac{1}{2m-2} \tan^{2m-2} x + \cdots + \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x + \ln|\sec x| + C
\end{aligned}
$$

$\left(I_1 = \int \tan x \, dx = \ln|\sec x| + C\right)$

§ 9 Definite Integration

9.1 Riemann Sum

Goal : Find the area of the region under the curve $y=f(x)$ over an interval $[a,b]$.

Wait ! We know what the area of a rectangle is.

However, what is the area of a region with a curved boundary ? (How to define ?)

Idea :

Approximate by rectangles !

A partition of the interval $[a,b]$ is a finite set $\{x_0, x_1, \cdots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b .$$

We denote $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \cdot\ , n$.

Then, we choose points, $c_1, c_2, \cdots, c_n$, called partition points so that

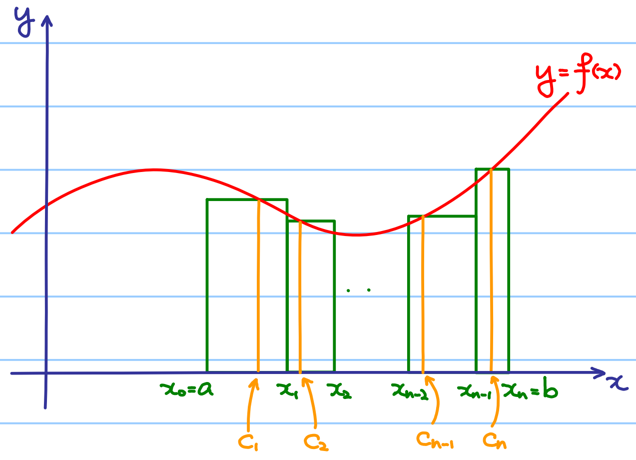
$$x_{k-1} \leq c_k \leq x_k \quad \text{for } k = 1, 2, \cdots, n .$$

Definition 9.1.1

Let $f : [a,b] \to \mathbb{R}$. The Riemann sum is defined by $\sum_{k=1}^{n} f(c_k) \Delta x_k$.

In particular, if $x_{k-1} = c_k$, the sum is called the left Riemann sum ;

if $c_k = x_k$, the sum is called the right Riemann sum.



For the k-th rectangle :

$$\underbrace{f(c_k)}_{\text{height}} \times \underbrace{\Delta x_k}_{\text{width}} = \text{area of the k-th rectangle}$$

Example 9.1.1

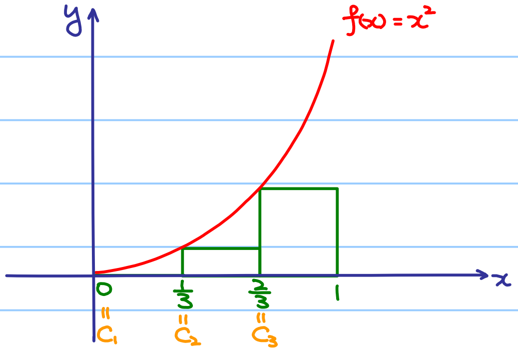
Let $f(x) = x^2$.

Approximate area under $f(x)$ over $[0,1]$

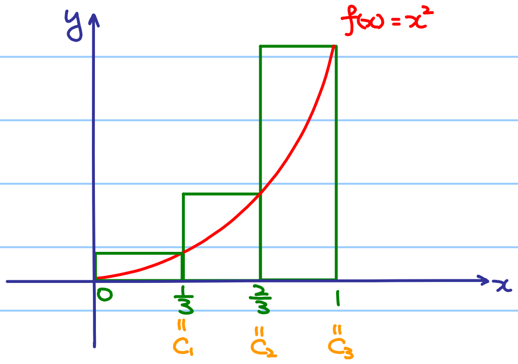
with 3 even partitions : $0 < \frac{1}{3} < \frac{2}{3} < 1$ $(x_0 = 0 , x_1 = \frac{1}{3} , x_2 = \frac{2}{3} , x_3 = 1)$

Riemann Sum :

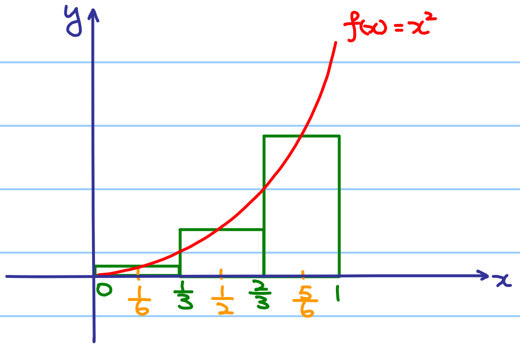
Left sum : $c_1 = 0 , c_2 = \frac{1}{3} , c_3 = \frac{2}{3}$ area $\approx 0^2 \cdot \frac{1}{3} + (\frac{1}{3})^2 \cdot \frac{1}{3} + (\frac{2}{3})^2 \cdot \frac{1}{3} = \frac{5}{27}$



Right sum : $c_1 = \frac{1}{3} , c_2 = \frac{2}{3} , c_3 = 1$ area $\approx (\frac{1}{3})^2 \cdot \frac{1}{3} + (\frac{2}{3})^2 \cdot \frac{1}{3} + 1^2 \ \frac{1}{3} = \frac{14}{27}$



Mid-pt sum : $c_1 = \frac{1}{6} , c_2 = \frac{1}{2} , c_3 = \frac{5}{6}$ area $\approx (\frac{1}{6})^2 \cdot \frac{1}{3} + (\frac{1}{2})^2 \cdot \frac{1}{3} + (\frac{5}{6})^2 \cdot \frac{1}{3} = \frac{35}{108}$



Idea :

Increasing n (number of rectangles) ⇒ Better approximation

Theorem 9.1.1

If $f:[a,b]\to\mathbb{R}$ is continuous (or piecewise continuous),

and $\Delta x_k = \Delta x = \frac{b-a}{n}$ for $k=1,2,\cdots,n$ (even partition), $x_k = a+k\Delta x$ for $k=0,1,2,\cdots,n$,

then no matter how c_k's are chosen, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} f(c_k)\Delta x$ is always the same

The area under $f(x)$ over $[a,b]$ is defined to be this number,

which is denoted by $\int_a^b f(x)\,dx$.

(Remark: Nothing related to indefinite integration so far!)

In fact, computation of the area is not relying on the above theorem,

but the fundamental theorem of calculus (Later!)

9.2 Rules for Definite Integration

Theorem 9.2.1

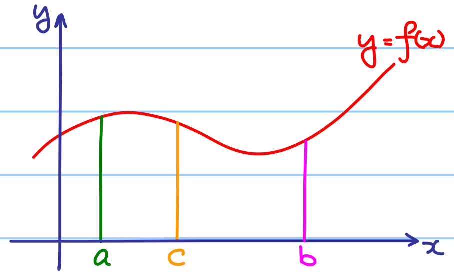
Let $f,g:\mathbb{R}\to\mathbb{R}$ be continuous (or piecewise continuous) functions.

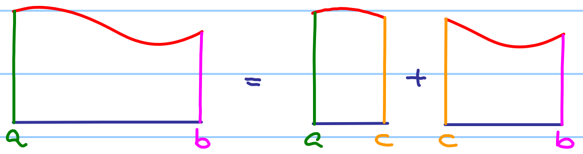
Suppose $a\le b$.

1) If k is a constant, $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
2) $\int_a^b f(x)\pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
3) $\int_a^a f(x)\,dx = 0$
4) $\int_b^a f(x)\,dx$ is defined to be $-\int_a^b f(x)\,dx$ (reverse direction)
5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any $c\in\mathbb{R}$ (subdivision)

Geometric meaning of (5) :

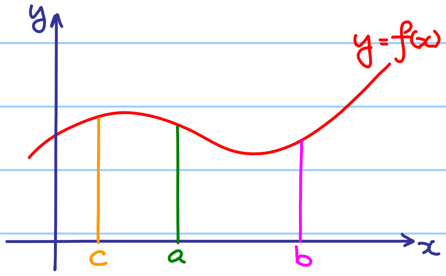
If $a \leq c \leq b$,

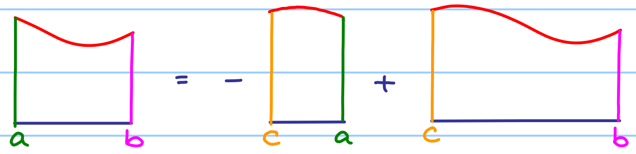




$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If $c < a \leq b$,





$$\int_a^b f(x)dx = \underbrace{\int_a^c f(x)dx}_{\overset{\parallel}{-\int_c^a f(x)dx}} + \int_c^b f(x)dx$$

Exercise :

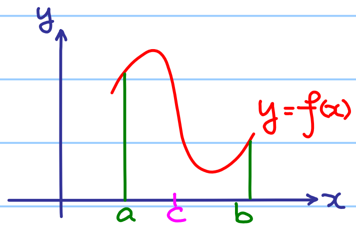
Think : Why (5) is true if $a \leq b < c$

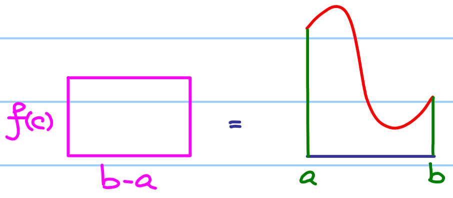
9.3 Fundamental Theorem of Calculus

Theorem 9.3.1 (Mean Value Theorem for integrals)

Let $f : [a,b] \to \mathbb{R}$ be a continuous function

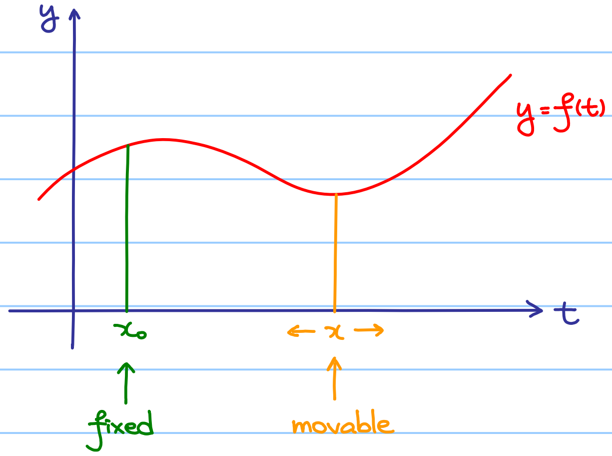
Then , there exists $c \in [a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$





Preparation :

Let $f(t)$, $t\in\mathbb{R}$, be a continuous function.



1) $\int_{x_0}^{x} f(t)dt$ is well defined for all $x\in\mathbb{R}$

2) What is a function? Roughly speaking, input x, output y.

Now, construct a new function $F(x)$ defined by

$$F(x) = \text{Area under the curve } y=f(t) \text{ over } [x_0, x]$$
$$= \int_{x_0}^{x} f(t)dt$$

Theorem 9.3.2 (Fundamental Theorem of Calculus)

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function and let $x_0\in\mathbb{R}$.

Suppose $F(x)$ is a function defined by

$$F(x) = \int_{x_0}^{x} f(t)dt ,$$

then $F(x)$ is a differentiable function and $F'(x) = f(x)$.

(i.e. $F(x)$ is an antiderivative of $f(x)$.)

1) Direct consequence : $\int_a^b f(x)dx = \int_{x_0}^{b} f(x)dx - \int_{x_0}^{a} f(x)dx$

$$= F(b) - F(a)$$

i.e. if we know how to compute antiderivative of $f(x)$,

then we know how to find $\int_a^b f(t)dt$.

2) Wait! Antiderivative of $f(x)$ is NOT unique, but unique up to a constant.

Which one should we pick?

If $\tilde{F}(x)$ is another antiderivative of $f(x)$, then $\tilde{F}(x) = F(x) + C$, where C is a constant.

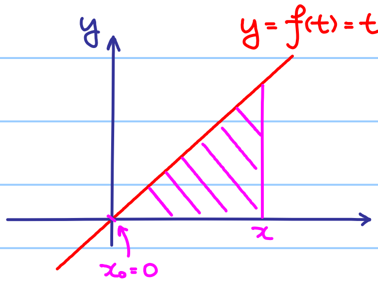
$$\begin{aligned}\tilde{F}(b) - \tilde{F}(a) &= (F(b)+C) - (F(a)+C) \\ &= F(b) - F(a) \\ &= \int_a^b f(x)\,dx .\end{aligned}$$

Therefore, we can pick anyone!

Example 9.3.1 (Verification of Fundamental Theorem of Calculus)

Let $f(t) = t$, $x_0 = 0$

$f(x) = x$

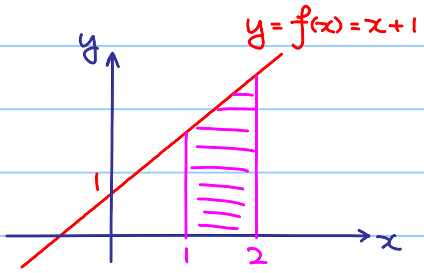


$$\begin{aligned}F(x) &= \int_{x_0}^{x} f(t)\,dt \\ &= \text{Area of the shaded triangle} \\ &= \frac{1}{2}x^2\end{aligned}$$

Note: We have $F'(x) = f(x)$.

Example 9.3.2

Let $f(x) = x+1$



Antiderivative of $f(x) = \int x+1\,dx = \frac{x^2}{2} + x + C$

Choose $C = 0$, let $F(x) = \frac{x^2}{2} + x$

$$\begin{aligned}\text{Area of the shaded region} = \int_1^2 f(x)\,dx &= F(2) - F(1) \\ &= 4 - \frac{3}{2} \\ &= \frac{5}{2}\end{aligned}$$

What we write:

$$\begin{aligned}\int_1^2 f(x)\,dx &= \left[\frac{x^2}{2} + x\right]_1^2 \\ &= \underbrace{\left(\frac{2^2}{2}+2\right)}_{F(2)} - \underbrace{\left(\frac{1^2}{2}+1\right)}_{F(1)} = 4 - \frac{3}{2} = \frac{5}{2}\end{aligned}$$

Example 9.3.3

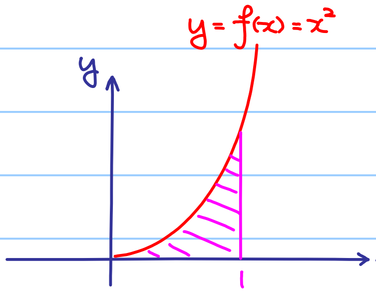
Let $f(x) = x^2$

Area of the shaded region $= \int_0^1 f(x)\,dx$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1^3}{3}\right) - \left(\frac{0^3}{3}\right)$$

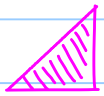
$$= \frac{1}{3}$$

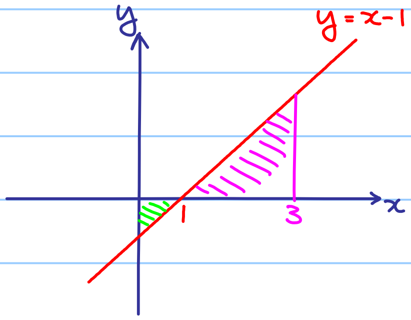


Example 9.3.4 (NOT area, but signed area)

$$\int_0^1 x-1\,dx = \left[\frac{x^2}{2}-x\right]_0^1 = -\frac{1}{2}$$

$$\int_1^3 x-1\,dx = \left[\frac{x^2}{2}-x\right]_1^3 = 2$$

$\int_0^3 x-1\,dx = \left[\frac{x^2}{2}-x\right]_0^3 = \frac{3}{2}$  –  (Cancellation)



Example 9.3.5

Find $\int_{-2}^3 |x-1|\,dx$

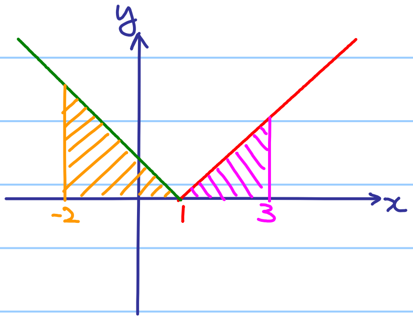
Recall : We can rewrite $|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -(x-1) & \text{if } x < -1 \end{cases}$

$$\int_{-2}^3 |x-1|\,dx = \int_{-2}^1 |x-1|\,dx + \int_1^3 |x-1|\,dx$$

$$= \int_{-2}^1 -(x-1)\,dx + \int_1^3 x-1\,dx$$

Exercise :

$$= \frac{9}{2} + 2 = \frac{13}{2}$$



Example 9.3.6

Find $\frac{dF}{dx}$ if

a) $F(x) = \int_0^x e^{\cos t}\,dt$, b) $F(x) = \int_0^{x^2} e^{\cos t}\,dt$, c) $F(x) = \int_x^{x^2} e^{\cos t}\,dt$

a) $\frac{dF}{dx} = e^{\cos x}$ (Directly from the Fundamental Theorem of Calculus , $f(x) = e^{\cos x}$)

b) $\frac{dF}{dx} = \frac{d}{dx^2} \int_0^{x^2} e^{\cos t}\,dt \;\; \frac{dx^2}{dx}$ (Chain rule)

$= e^{\cos x^2} \cdot 2x$

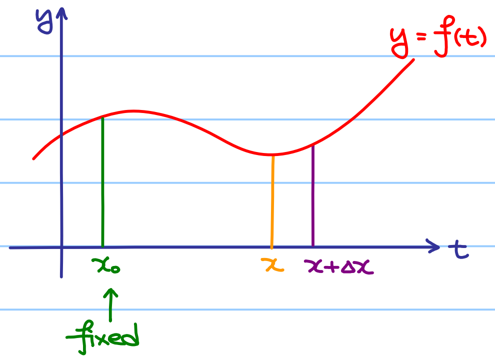
$= 2x e^{\cos x^2}$

c) $\frac{dF}{dx} = \frac{d}{dx} \int_0^{x^2} e^{\cos t}\,dt - \frac{d}{dx} \int_0^{x} e^{\cos t}\,dt$

$= 2xe^{\cos x^2} - e^{\cos x}$

Proof of the Fundamental Theorem of Calculus :

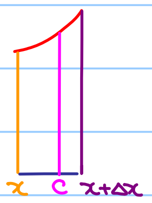
Claim : If $F(x) = \int_{x_0}^{x} f(t)\,dt$, $\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x)$, i.e. $F'(x) = f(x)$



$F(x+\Delta x) - F(x)$

$= \int_x^{x+\Delta x} f(t)\,dt$

= area of



$= f(c)\,\Delta x$ for some c between x and $x+\Delta x$

(Mean Value theorem for integrals)

$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f(c)\,\Delta x}{\Delta x}$

$= \lim_{\Delta x \to 0} f(c)$

$= \lim_{c \to x} f(c)$ (As Δx tends to 0 , c tends to x)

$= f(x)$ (By continunity of f)

$\therefore$ $F(x)$ is differentiable and $F'(x) = f(x)$.

9.4 Definite Integral Using Substitution

Theorem 9.4.1

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Example 9.4.1

Evaluate $\int_0^1 8x(x^2+1)\,dx$

$$\int_0^1 8x(x^2+1)\,dx$$
$$= \int_1^2 8u \cdot \frac{1}{2}du$$
$$= \int_1^2 4u\,du$$
$$= \left[2u^2\right]_1^2$$
$$= 6$$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

} Similar to indefinite integration

when $x=0$, $u=1$

$x=1$, $u=2$

} New! Don't forget!

Remark:

Some may write :

Still 0 and 1

$$\int_0^1 8x(x^2+1)\,dx = \int_0^1 4(x^2+1)\,d(x^2+1)$$
$$= \left[2(x^2+1)\right]_0^1$$
$$= 6$$

(as $d(x^2+1) = 2x\,dx$)

(Just the same result !)

Example 9.4.2

Evaluate $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx$$
$$= \int_1^2 \frac{1}{u}\,du$$
$$= \left[\ln u\right]_1^2$$
$$= \ln 2 - \cancelto{0}{\ln 1}$$
$$= \ln 2$$

Let $u = \ln x$

$du = \frac{1}{x}dx$

when $x=e$, $u=1$

$x=e^2$, $u=2$

9.5 Definite Integration Using Integration by Parts

Theorem 9.5.1

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx$$

Example 9.5.1

Evaluate $\int_1^e x\ln x\,dx$

$$\int_1^e x\ln x\,dx = \int_1^e \ln x \; d\left(\frac{x^2}{2}\right)$$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\,d\ln x$$

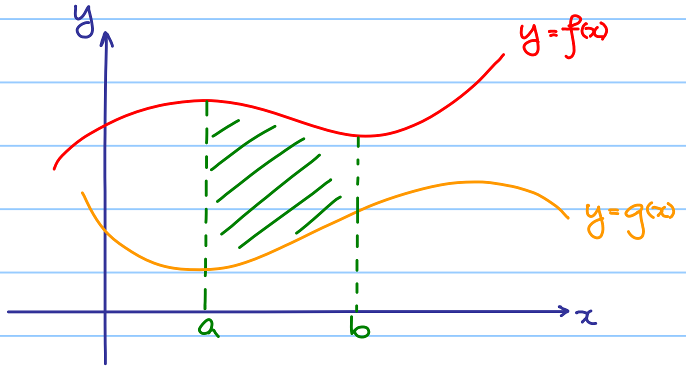
$$= \left(\frac{e^2}{2}\ln e - \cancelto{0}{\frac{1}{2}\ln 1}\right) - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$

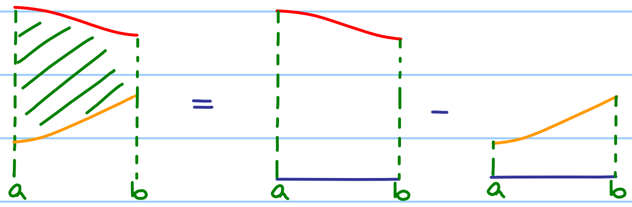
$$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

9.6 Area Between Curves



Area of shaded region = $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



Example 9.6.1

Find the area bounded by $y = x^2$ and $y = x^3$.

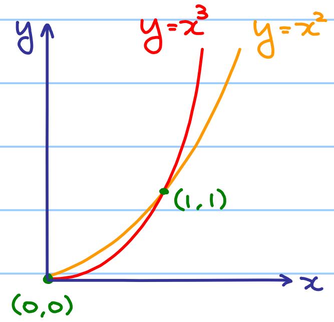
Step 1 : Solve $\begin{cases} y = x^2 \\ y = x^3 \end{cases}$

$$x^3 = x^2$$

$$x^2(x-1) = 0$$

$$x = 0 \quad \text{or} \quad 1$$

(Remark : No need to solve y)



Step 2 : Note when $0 \leq x \leq 1$, $x^3 \leq x^2$

$$\text{Area} = \int_0^1 x^2 - x^3\,dx$$

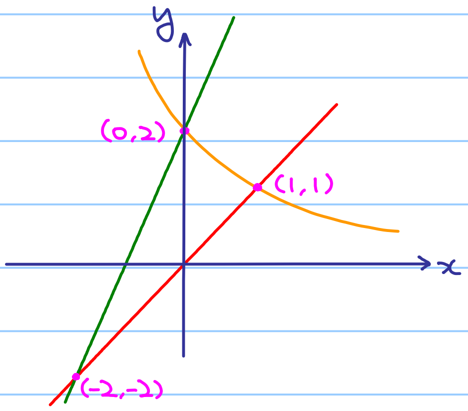
$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{12}$$

Example 9.6.2

Find the area bounded by

$y = f(x) = x$, $y = g(x) = \frac{2}{x+1}$ and $y = h(x) = 2x+2$



Area $= \int_{-2}^{0} h(x) - f(x) \, dx + \int_{0}^{1} g(x) - f(x) \, dx$

Exercise :

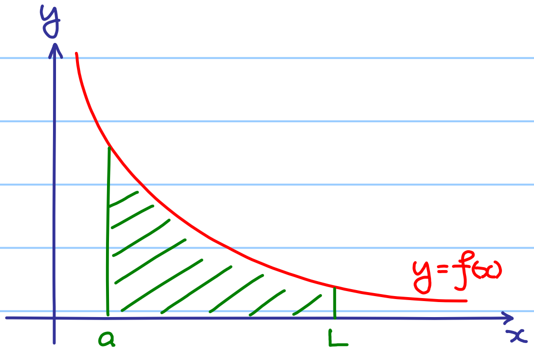
$= 2 + (-\frac{1}{2} + \ln 4)$

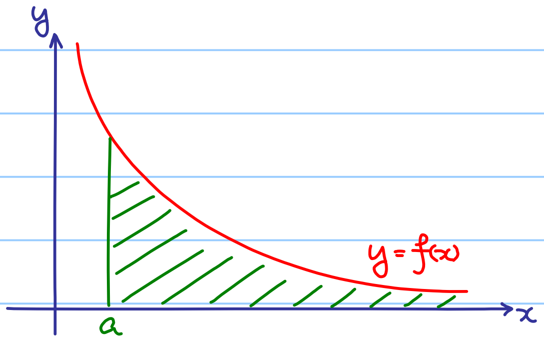
$= \frac{3}{2} + \ln 4$

9.7 Improper Integrals

Question : Find the area of the unbounded region ?

Idea :



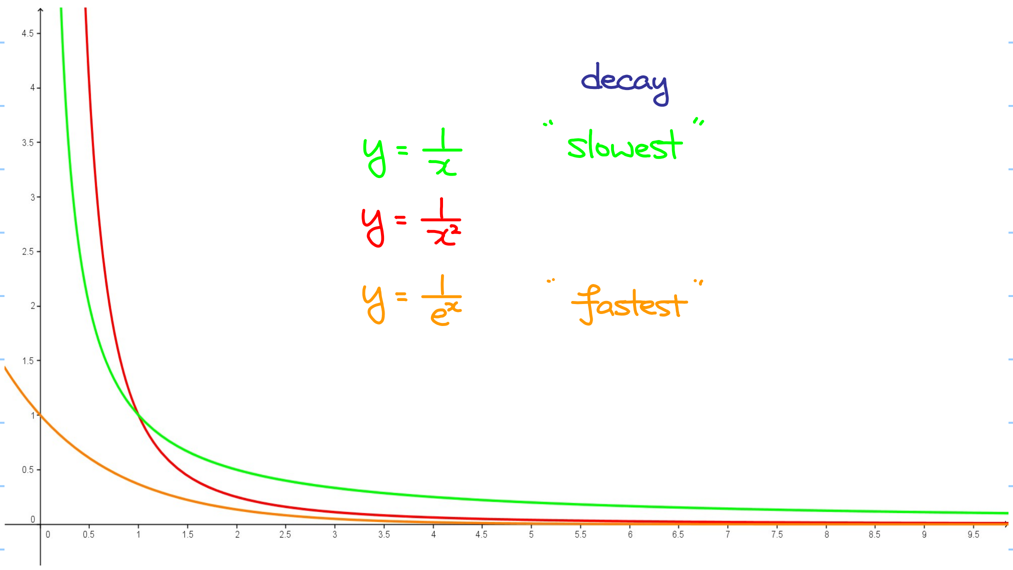


$\int_a^L f(x) dx$ ⇝ Area of the unbounded region

$= \lim_{L \to +\infty} \int_a^L f(x) dx$ (if it exists)

We denote it by $\int_a^{+\infty} f(x) \, dx$

Example 9.7.1



decay

$y = \frac{1}{x}$ "slowest"

$y = \frac{1}{x^2}$

$y = \frac{1}{e^x}$ "fastest"

① $\lim_{L\to+\infty}\int_1^L \frac{1}{x}\,dx = \lim_{L\to+\infty}\left[\ln x\right]_1^L = \lim_{L\to+\infty} \ln L = +\infty$ (i.e. limit does NOT exist)

② $\lim_{L\to+\infty}\int_1^L \frac{1}{x^2}\,dx = \lim_{L\to+\infty}\left[-\frac{1}{x}\right]_1^L = \lim_{L\to+\infty} 1-\frac{1}{L} = 1$

③ $\lim_{L\to+\infty}\int_1^L \frac{1}{e^x}\,dx = \lim_{L\to+\infty}\left[-\frac{1}{e^x}\right]_1^L = \lim_{L\to+\infty} -\frac{1}{e^L}+\frac{1}{e} = \frac{1}{e}$

Observation : $\lim_{x\to+\infty} f(x) = 0$ does NOT guarantee $\lim_{L\to+\infty}\int_a^L f(x)\,dx$ exists.

Example 9.7.2

Find $\int_0^{+\infty} \frac{1}{(x+1)(3x+2)}\,dx$

Note : $(x+1)(3x+2)$ is a polynomial of degree 2.

$\frac{1}{(x+1)(3x+2)}$ decays as "fast" as $\frac{1}{x^2}$.

$$\lim_{L\to+\infty}\int_0^L \frac{1}{(x+1)(3x+2)}\,dx = \lim_{L\to+\infty}\int_0^L \frac{-1}{x+1}+\frac{3}{3x+2}\,dx$$

$$= \lim_{L\to+\infty}\left[\ln|x+1| + \ln|3x+2|\right]_0^L$$

$$= \lim_{L\to+\infty} \ln\left|\frac{3L+2}{L+1}\right| - \ln 2$$

$$= \ln 3 - \ln 2$$

Example 9.7.3

Find $\int_0^{+\infty} xe^{-2x}dx$

$$\lim_{L\to+\infty}\int_0^L xe^{-2x}dx$$

$$= \lim_{L\to+\infty}\int_0^L x\,d(-\tfrac{1}{2}e^{-2x})$$

$$= \lim_{L\to+\infty}[-\tfrac{1}{2}xe^{-2x}]_0^L - \int_0^L -\tfrac{1}{2}e^{-2x}dx$$

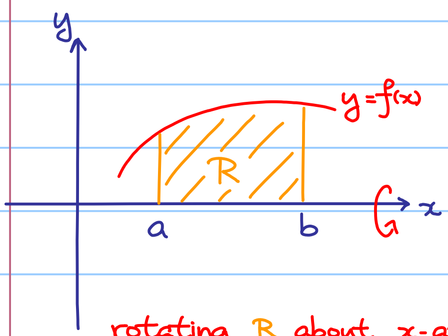
$$= \lim_{L\to+\infty}[-\tfrac{1}{2}xe^{-2x}]_0^L + [-\tfrac{1}{4}e^{-2x}]_0^L$$

tend to 0 when $L\to+\infty$

$$= \lim_{L\to+\infty} -\tfrac{1}{2}Le^{-2L} - \tfrac{1}{4}e^{-2L} + \tfrac{1}{4}$$

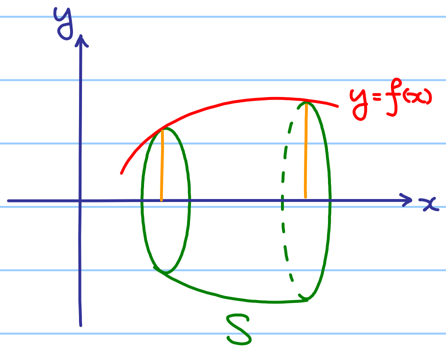
$$= \tfrac{1}{4}$$

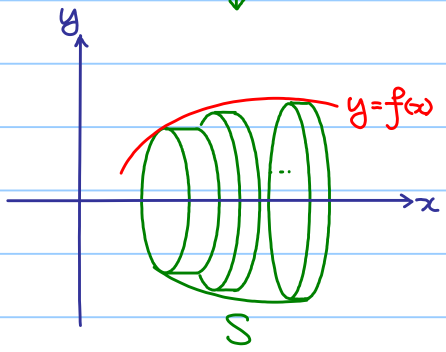
9.8 Solids of Revolution and Disk Method

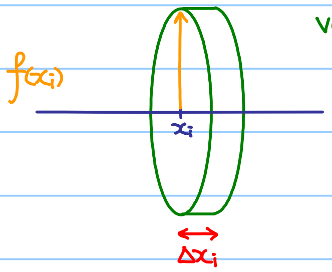


rotating R about x-axis

gives a solid S







volume $= \pi[f(x_i)]^2\Delta x_i$

Volume of S $= \lim_{n\to\infty}\sum_{i=1}^{n}\pi[f(x_i)]^2\Delta x_i$

$$= \int_a^b \pi[f(x)]^2dx$$

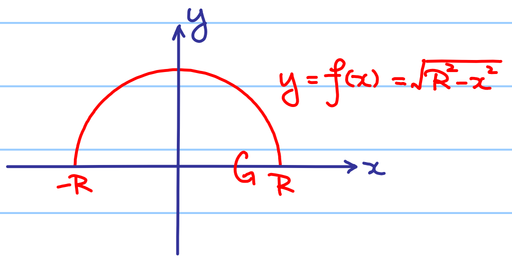
$$= \pi\int_a^b f(x)^2dx$$

Approximate volume of S

by solid disks.

Example 9.8.1

Let $f(x) = \sqrt{R^2 - x^2}$ for $-R \le x \le R$



semi-circle

Solid S = sphere

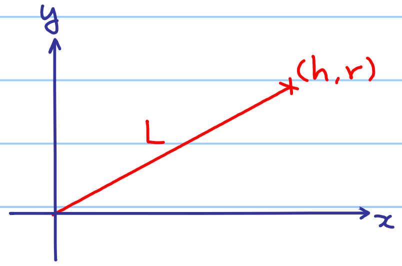
volume of S $= \pi \int_{-R}^{R} [f(x)]^2 dx$

$= \pi \int_{-R}^{R} R^2 - x^2 dx$

$= \pi \left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R}$

$= \frac{4}{3} \pi R^3$ (formula in secondary school)

Exercise 9.8.1



a) Find the equation of the straight line L.

b) What is the solid S generated by rotating L about the x-axis?

c) Volume of S = ?

Ans. a) $y = \frac{r}{h} x$

b) a cone with height = h, base radius = r

c) $\pi \int_0^h (\frac{r}{h}x)^2 dx = \frac{1}{3} \pi r^2 h$

Example 9.8 2

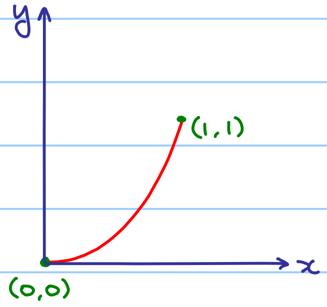
Let C be a curve given by $y = x^2$ for $0 \leq x \leq 1$.

Find the volume of the solid generated by rotating C about the axis:

a) $y = -1$

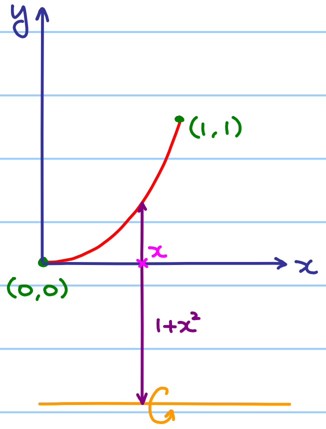
b) the y-axis

c) $x = -1$



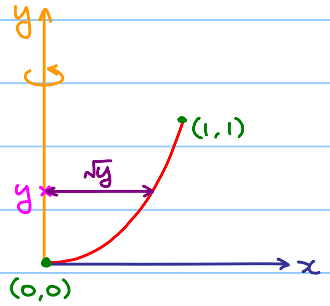
a) $y = -1$

$$\text{Volume} = \int_0^1 \pi(1+x^2)^2 \, dx$$
$$= \pi \int_0^1 x^4 + 2x^2 + 1 \, dx$$
$$= \pi \left[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x\right]_0^1$$
$$= \tfrac{28}{15}\pi$$



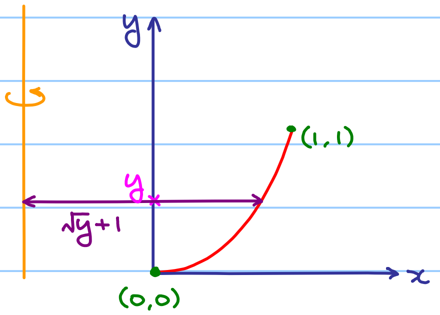
b) the y-axis

$$\text{Volume} = \int_0^1 \pi(\sqrt{y})^2 \, dy$$
$$= \pi \int_0^1 y \, dy$$
$$= \pi \left[\tfrac{1}{2}y^2\right]_0^1$$
$$= \tfrac{1}{2}\pi$$



c) $x = -1$

$$\text{Volume} = \int_0^1 \pi(\sqrt{y}+1)^2 \, dy$$
$$= \pi \int_0^1 y + 2\sqrt{y} + 1 \, dy$$
$$= \pi \left[\tfrac{1}{2}y^2 + \tfrac{4}{3}y^{\frac{3}{2}} + y\right]_0^1$$
$$= \tfrac{17}{6}\pi$$



§10 Power Series and Talyor Series

10.1 Power Series

Definition 10.1.1

A power series is an infinite series of the form:

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0+a_1(x-c)+a_2(x-c)^2+\cdots$$

where c is called the center.

Example 10 1 1

$f(x)=\sum_{n=0}^{\infty} x^n$ (power series centered at $x=0$, i.e. $c=0$)

(all a_n's equal to 1)

$$=1+x+x^2+x^3+\cdots$$

$$f\left(\tfrac{1}{2}\right)=1+\tfrac{1}{2}+\left(\tfrac{1}{2}\right)^2+\left(\tfrac{1}{2}\right)^3+\cdots \overset{\text{G.P.}}{=} \frac{1}{1-\frac{1}{2}}=2$$

However,

$f(2)=1+2+2^2+2^3+\cdots$ (does NOT converge)

Main question:

Find the possible value(s) of x such that

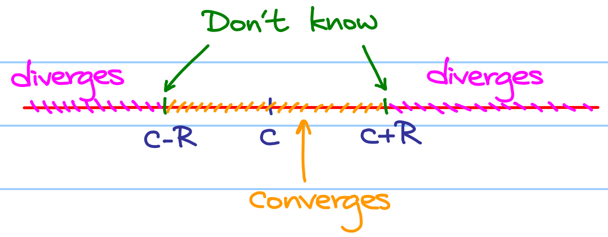
$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0+a_1(x-c)+a_2(x-c)^2+\cdots$ converges ?

Theorem 10.1.1

Let $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ if it exists or $+\infty$

Then $f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0+a_1(x-c)+a_2(x-c)^2+\cdots$ converges

when $c-R<x<c+R$, but diverges when $x<c-R$ or $x>c+R$



R is called the radius of convergence

Example 10.1.1 (Cont.)

$f(x) = \sum_{n=0}^{\infty} x^n$ (power series centered at $x=0$, i.e. $c=0$)

(all a_n's equal to 1)

$= 1 + x + x^2 + x^3 + \cdots$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{1} = 1$$

$\therefore R = 1$, $f(x) = \sum_{n=0}^{\infty} x^n$ converges when $-1 < x < 1$.

Example 10.1.2

$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ (power series centered at $x=0$, i.e. $c=0$)

$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ ($a_n = \frac{1}{n!}$)

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} n+1 = +\infty$$

$\therefore R = +\infty$ (Convention), $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all $x \in \mathbb{R}$

Example 10.1.3

$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{n}{4^n} (x+2)^n$ (power series centered at $x=0$, i.e. $c=-2$)

($a_n = (-1)^n \frac{n^2}{4^n}$)

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{(-1)^n \frac{n^2}{4^n}}{(-1)^{n+1} \frac{(n+1)^2}{4^{n+1}}} \right| = \lim_{n\to\infty} \frac{4n^2}{(n+1)^2} = 4$$

$\therefore R = 4$, $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{n}{4^n} (x+2)^n$ converges for all $\overset{c-R}{\overset{\downarrow}{-6}} < x < \overset{c+R}{\overset{\downarrow}{2}}$

10.2 Talyor Polynomials

Definition 10.2.1

Let $f(x)$ be a function with derivatives of all orders on an open interval I, and $c \in I$.

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

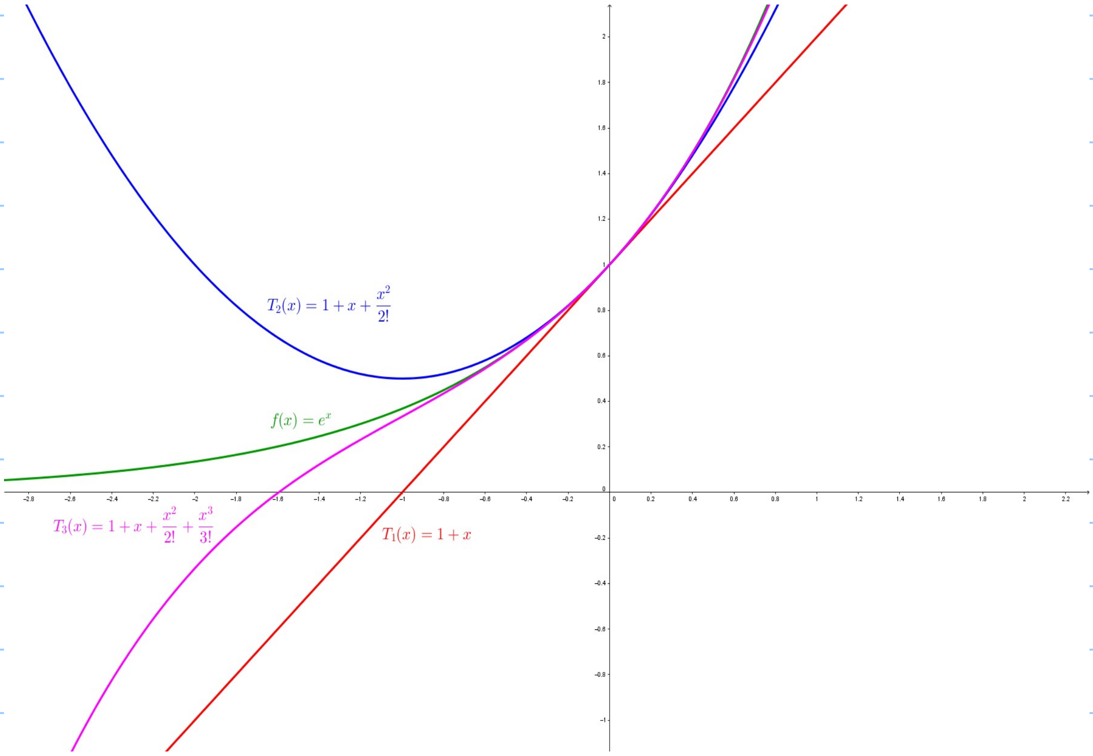
is called the Talyor polynomial of order n generated by f at c.

Example 10.2.1

Let $f(x) = e^x$, find the Talyor polynomials $T_n(x)$ generated by f at $x=0$.

Note : $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = 1$ for $k = 0, 1, 2, \cdots, n$

$$\therefore \quad T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$
$$= 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$$
$$= \sum_{k=0}^{n} \frac{1}{k!}x^k$$



Note that:

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \Rightarrow \quad T_n(c) = f(c)$$

$$T_n'(x) = f'(c) + f''(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1} \quad \Rightarrow \quad T_n'(c) = f'(c)$$

$$T_n''(x) = f''(c) + \cdots + \frac{f^{(n)}(c)}{(n-2)!}(x-c)^{n-2} \quad \Rightarrow \quad T_n''(c) = f''(c)$$

$$\vdots$$

$$T_n^{(n)}(x) = f^{(n)}(c) \quad \Rightarrow \quad T_n^{(n)}(c) = f^{(n)}(c)$$

.. $T_n(x)$ approximates $f(x)$ around the point c in a sense that

$$\left.\begin{array}{l} T_n(c) = f(c) \\ T_n'(c) = f'(c) \\ T_n''(c) = f''(c) \\ \\ T_n^{(n)}(c) = f^{(n)}(c) \end{array}\right\} \quad T_n(x) \text{ and } f(x) \text{ agree at the point } c \text{ up to } n\text{-th derivative}$$

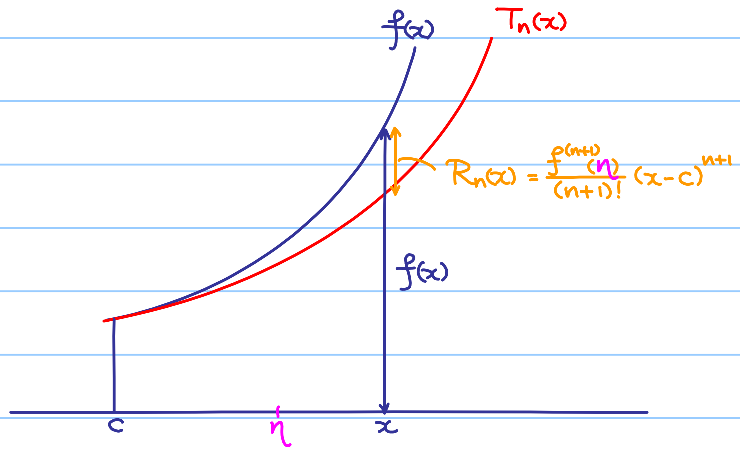
10.3 Talyor's Theorem

Theorem 10.3.1 (Talyor's Theorem)

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between x and c, $f^{(n)}$ is differentiable on the open interval between x and c, then there exists η between x and c such that

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-c)^{n+1} \\ &= \sum_{r=0}^{n} \frac{f^{(r)}(c)}{r!}(x-c)^r + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-c)^{n+1} \\ &= T_n(x) + R_n(x) \end{aligned}$$

i.e. Approximate $f(x)$ by $T_n(x)$, then the error can be expressed as $R_n(x) = \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-c)^{n+1}$



i.e. the error can be described by the (n+1)-th derivative of f.

Example 10.3.1

Approximate cos 0.1

Let $f(x) = \cos x$,

$T_5(x) = T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ Talyor polynomials generated by f at $x=0$.

$\cos 0.1 = f(0.1) \approx T_5(0.1) = 0.995004166\cdots$

By Talyor's Theorem $f(0.1) = T_5(0.1) + \frac{f^{(6)}(\eta)}{6!}(0.1)^6$ $\quad \eta \in (0, 0.1)$

Absolute Error $= \left|\frac{f^{(6)}(\eta)}{6!}(0.1)^6\right|$

$\leq \frac{1}{6!}(0.1)^6 \approx 1.38 \times 10^{-9}$

Very small.

Note : $f^{(6)}(x) = -\cos x$

$\Rightarrow |f^{(6)}(\eta)| \leq 1$

Idea:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-c)^{n+1}$$

$$= T_n(x) + R_n(x)$$

$R_n(x) = f(x) - T_n(x)$

If $\lim\limits_{n\to\infty} R_n(x) = 0$, i.e. error tends to 0, then

$$\lim_{n\to\infty} f(x) - T_n(x) = 0$$

$$f(x) = \lim_{n\to\infty} T_n(x)$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{f^{(r)}(c)}{r!}(x-c)^r$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Definition 10.3.1

Suppose that $f^{(n)}(c)$ exist for all $n = 0, 1, 2, \cdots$.

Taylor Series generated by f at $x=c$ is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

In particular, if $c=0$, the series is called a Maclaurin series.

Technical problems:

1) Convergence of Taylor series ?

In fact, a Talyor series is a power series, radius of convergence = ?

2) Converges to $f(x)$?

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} f(x) - T_n(x) \overset{?}{=} 0$$

Frequently used Talyor series:

1) $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$, $\quad |x| < 1$

2) $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$, $\quad |x| < 1$

3) $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\quad \forall x \in \mathbb{R}$

4) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\quad \forall x \in \mathbb{R}$

5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $\quad \forall x \in \mathbb{R}$

6) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, $\quad \forall\ -1 < x \le 1$

Remark. They are Maclaurin series in fact.

Example 10.3.2

Find the Talyor series generated from $f(x) = e^{2x}$ at $x = 1$

Note. $f(1) = e^2$

$f'(x) = 2e^{2x} \Rightarrow f'(1) = 2e^2$

$f''(x) = 2^2 e^{2x} \Rightarrow f''(1) = 2^2 e^2$

$\vdots$

$f^{(r)}(x) = 2^r e^{2x} \Rightarrow f^{(r)}(1) = 2^r e^2$

Talyor series generated from $f(x) = e^{2x}$ at $x = 1$.

$$\sum_{r=0}^{\infty} \frac{f^{(r)}(1)}{r!} (x-1)^r = \sum_{r=0}^{\infty} \frac{2^r e^2}{r!} (x-1)^r$$

10.4 Operations of Taylor Series

From the above frequently used Talyor series, we can find the Talyor series of more complicated functions without starting from definition 10.3.1 .

Example 10.4.1

Recall:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n},$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad ,$$

1) (Addition)

$$\cos x + \sin x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

2) (Subtraction)

$$\cos x - \sin x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

3) (Product)

$$\begin{aligned}
\cos x \sin x &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(-\frac{x^3}{3!}\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(\frac{x^5}{5!}\right) + \cdots \\
&= x \quad - \frac{x^3}{2!} \quad + \frac{x^5}{4!} \quad - \cdots \\
& - \frac{x^3}{3!} \quad + \frac{x^5}{2!3!} \quad - \cdots \\
&\phantom{= x \quad - \frac{x^3}{3!} \quad} + \frac{x^5}{5!} \quad - \cdots \\
&= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \cdots
\end{aligned}$$

4) (Composition)

$$\begin{aligned}
e^{\sin x} &= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^3 + \cdots \\
&= 1 + x + \frac{x^2}{2} + \cdots
\end{aligned}$$

5) (Division)

Let $\frac{\sin x}{\cos x} = a_0 + a_1 x + a_2 x^2 + \cdots$

$$\therefore \ \sin x = \cos x \ (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)$$

$$\begin{aligned} = a_0 + a_1 x + a_2 x^2 &+ a_3 x^3 + a_4 x^4 + \cdots \\ - \frac{a_0}{2!} x^2 &- \frac{a_1}{2!} x^3 - \frac{a_2}{2!} x^4 - \cdots \\ &+ \frac{a_0}{4!} x^4 + \cdots \end{aligned}$$

Compare coefficients of x^r for $r = 0, 1, 2, 3, 4, \cdots$:

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 - \frac{a_0}{2!} = 0 \\ a_3 - \frac{a_1}{2!} = -\frac{1}{3!} \\ a_4 - \frac{a_2}{2!} + \frac{a_0}{4!} = 0 \\ \quad \vdots \end{cases}$$

$\therefore \ a_0 = 0, \ a_1 = 1, \ a_2 = 0, \ a_3 = \frac{1}{3}, \ a_4 = 0, \cdots$

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{1}{3} x^3 + \cdots$$

Example 10.4.2

Recall: $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$

Find the Talyor series generated by $\frac{1}{(1+x)^2}$ at $x=0$ by considering $\frac{d}{dx} \frac{1}{1+x} = -\frac{1}{(1+x)^2}$.

6) (Differentiation)

$$\frac{d}{dx} \frac{1}{1+x} = \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right]$$

$$\begin{aligned} -\frac{1}{(1+x)^2} &= \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}(x^n) \\ &= \sum_{n=0}^{\infty} (-1)^n n x^{n-1} \end{aligned}$$

$$\begin{aligned} \frac{1}{(1+x)^2} &= \sum_{n=0}^{\infty} (-1)^{n-1} n x^{n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1} \qquad (\because \text{ when } n=0, \ (-1)^{n-1} n x^{n-1} = 0) \end{aligned}$$

Find the Talyor series generated by $\tan^{-1}x$ at $x=0$ by considering $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$.

7) (Integration)

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int \frac{1}{1+x^2} dx = \sum_{n=0}^{\infty} \left((-1)^n \int x^{2n} dx \right)$$

$$\tan^{-1}x = C + \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

put $x=0$,

$$\tan^{-1}(0) = C + \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} (0)^{2n+1}$$

$$C = 0$$

$$\therefore \tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$